A Parallel Partial Solution Variant of Cyclic Reduction and Applications

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Abstract

The partial solution variant of cyclic reduction (PSCR) method is a fast direct solver for systems of linear equations with tensor product matrices. It was introduced by P.S. Vassilevski and Yuri Kuznetsov during the first half of 1980s [9, 6]. The coefficient matrix has the tensor product form

$$\boldsymbol{A}_1 \otimes \boldsymbol{M}_2 + \boldsymbol{M}_1 \otimes \boldsymbol{A}_2$$

or

$$\boldsymbol{A}_1 \otimes \boldsymbol{A}_2 \otimes \boldsymbol{M}_3 + \boldsymbol{A}_1 \otimes \boldsymbol{M}_2 \otimes \boldsymbol{A}_3 + \boldsymbol{M}_1 \otimes \boldsymbol{A}_2 \otimes \boldsymbol{A}_3,$$

where $\boldsymbol{A}_i$ and $\boldsymbol{M}_i$ are banded matrices with a narrow bandwidth. In most typical cases, $\boldsymbol{A}_i$ are tridiagonal and $\boldsymbol{M}_i$ are diagonal or tridiagonal. The computational complexity of the method is $\mathcal{O}(N \log N)$ and $\mathcal{O}(N \log^2 N)$ floating point operations for the above forms, respectively. Finite difference and finite element discretizations of separable operators on orthogonal grids/meshes result matrices having the above tensor product forms. Usual examples are the Poisson equation and the Helmholtz equation. The grids can be nonuniform and the boundary conditions can be also of Robin/absorbing type [2, 7].

The basic idea of the cyclic reduction methods is reduce equations from the system by eliminating unknowns on given grid lines at each level. This results Schur complement problems for remaining unknowns. The classical cyclic reduction (CR) method solves these problems using a rational matrix polynomials. The PSCR method uses the partial solution technique for these problems [7, 8] which is based on the following two observations: The matrices $\boldsymbol{A}_1$ and $\boldsymbol{M}_1$ can be diagonalized using the eigenvectors $\boldsymbol{W}$ of the generalized eigenvalue problem $\boldsymbol{A}_1\boldsymbol{W} = \boldsymbol{M}_1\boldsymbol{W}\boldsymbol{\Lambda}$. Taking advantage of the sparsity of resulting problems the transformations with the eigenvectors are not too expensive. There are two clear benefits to use the partial solution: The diagonalization by eigenvectors offers a natural way to make a parallel implementation of the PSCR method while the Schur complement problems in the CR cannot be solved in parallel in all levels. The partial solution makes it possible to reduce the number of equations by a larger factor than two while in the CR the reduction is always by factor of two. This leads to computational savings due to fewer levels. For example, our PDC2D implementation uses factor of four reductions [7]. The basic algorithm of the PSCR method will be explained.

The parallelization of the PSCR method is described using MPI communication [7]. The behavior of the solver is studied in many different parallel environments. In homogeneous parallel computers like Cray T3E and SGI Origin, the scalability has be demonstrated to be good up to 512 processors [1, 7]. On the other hand in metacomputing environments, the latency in communication can be too high and the bandwidth can be too limited leading to deteriorated speedups [1]. In such cases, the solver can be combined with a direct approximate Aitken-Schwarz domain decomposition which reduces the number of communications and the amount of data transfered over slower links [1].

The PSCR method has been used for many different kinds of applications including: acoustic scattering, electromagnetic scattering, elasticity, (full) potential flows, Navier-Stokes flows, multiphase flows in porous media, combustion problems, modeling tumor growth, shape and material optimizations. As an example of the applications, large scale acoustic scattering problems with more than a billion unknowns are shown [3, 4, 5].

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