Convergence of iterative solvers for the method coupling Finite elements and integral representation

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1 Introduction

The method coupling finite elements and integral representation (cf.[2]) allows to solve exterior problems, thanks to an exact boundary condition on a fictitious boundary, which delimitate the computational domain. This boundary condition is non local, the associated algebraic system is then not sparse. We study the convergence of an iterative solver based on a GMRES method, where at each iteration we have to inverse a sparse matrix. This method can be interpreted us a resolution of the Steklov-Poincaré interface equation, preconditioned by the matrix of the Alternating Schwarz method. We prove that GMRES algorithm converges, and compare this approach to other algorithms, based on domain decomposition methods.

2 The Scattering acoustic problem

Let Ω_i be a bounded obstacle (in $\mathbb{R}^3$) with a regular boundary Γ and Ω_e be its unbounded complementary. The Helmholtz problem is a fair modeling of a scattered acoustic wave propagating through Ω_e; it consists in *finding u such that*

$$\Delta u + \kappa^2 u = 0 \text{ in } \Omega_e, \qquad \partial_n u = f \text{ on } \Gamma, \tag{1}$$

$$(\frac{x}{|x|}.\nabla - i\kappa)u = e^{i\kappa|x|}O(\frac{1}{|x|^2}) \qquad x \in V_\infty, \tag{2}$$

where κ is the wave number, V_∞ is a neighbourhood of infinity. The last condition represents the Sommerfeld radiation condition.

To make of the boundary value problem (2) accessible to scientific computing, we may resort to the integral equation method, the efficicency of which has

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been investigated by several authors (cf.[3]). An alternative consists in using a coupled method which combines finite volume elements and integral representations (cf.[2]). This approach takes profit of the advantages of both techniques namely the possibility of truncating the domain of calculation –avoiding boundary finite elements– and the attractive properties of the approximation by finite elements of classical variational formulations. The idea simply amounts to introduce a fictitious boundary Σ surrounding the obstacle. The Helmholtz problem is posed in the truncated domain Ω_c (delimited by Γ and Σ). On the boundary Γ is set a natural Neumann condition whereas a non-standard outgoing condition using the integral formula is prescribed on Σ,

$$(\partial_n - i\kappa)u(x) = \int_\Gamma \big(u(y)\partial_n K(x-y) - f(y)K(x-y)\big)\mathrm{d}\gamma \quad \forall x \in \Sigma. \tag{3}$$

In (3), K is given by $K(x) = (\frac{x}{|x|}.\nabla - i\kappa)G_\kappa(x)$, G_κ is the Green function defined by $G_\kappa(x) = -\frac{1}{4\pi}\frac{e^{i\kappa|x|}}{|x|}$. Observe that the integral representation is used only in Σ which avoids occurrences of any singularities. The non-local coupling enforced by the integral term generates some difficulties in its numerical treatment because it perturbs the sparse structure of the stiffness matrix (around the degrees of freedom located on the artifical boundary). Let $g = (\partial_n - i\kappa)u$ defined on Σ. Writing the Steklov-Poincaré interface equation, leads to introduce an equivalent problem defined on Σ, under the form:

$$(I - A)g = g_{inc}. \tag{4}$$

where A is a compact operator from $L^2(\Sigma)$ on $L^2(\Sigma)$ and g_{inc} is related to the incident field. Under its discretized form, we show that the matrix associated to the operator A has only a few eigenvalues, out of the unit disk of the complex plane and analyze the convergence of the GMRES algorithm. We compare this approach to the preconditioned matrix system obtained by the discretization of the variational formulation associeted to (2, 3) and also described in (cf.[1]). We compare the convergence speed of this method with an iterative algorithm based on a Alternating Schwarz method.

References

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