On finding approximate supernodes for an efficient ILU(k) factorization

PMAA'06

Rennes

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Projet ScAlApplix, INRIA UR Futurs

Motivation of this work

- A popular choice as an algebraic preconditioner is an ILU(k) preconditioner (level-of-fill based inc. facto.)

- BUT

➢ Parallelization is not easy

➢ Scalar formulation does not take advantage of superscalar effect (i.e. BLAS)
=> Usually a low value of fill is used (k=0 or k=1)

Motivation of this work

ILU + Krylov Methods

Based on scalar implementation

Difficult to parallelize (mostly DD + Schwartz additive => # of iterations depends on the number of processors)

Low memory consumption

Precision ~ 10^-5

Direct methods

BLAS3 (mostly DGEMM)
Thread/SMP, Load Balance…

Parallelization job is done (MUMPS, PASTIX, SUPERLU…)

High memory consumption : very large 3D problems are out of their league (100 millions unknowns)

Great precision ~ 10^-18

We want a trade-off !

Motivation of this work

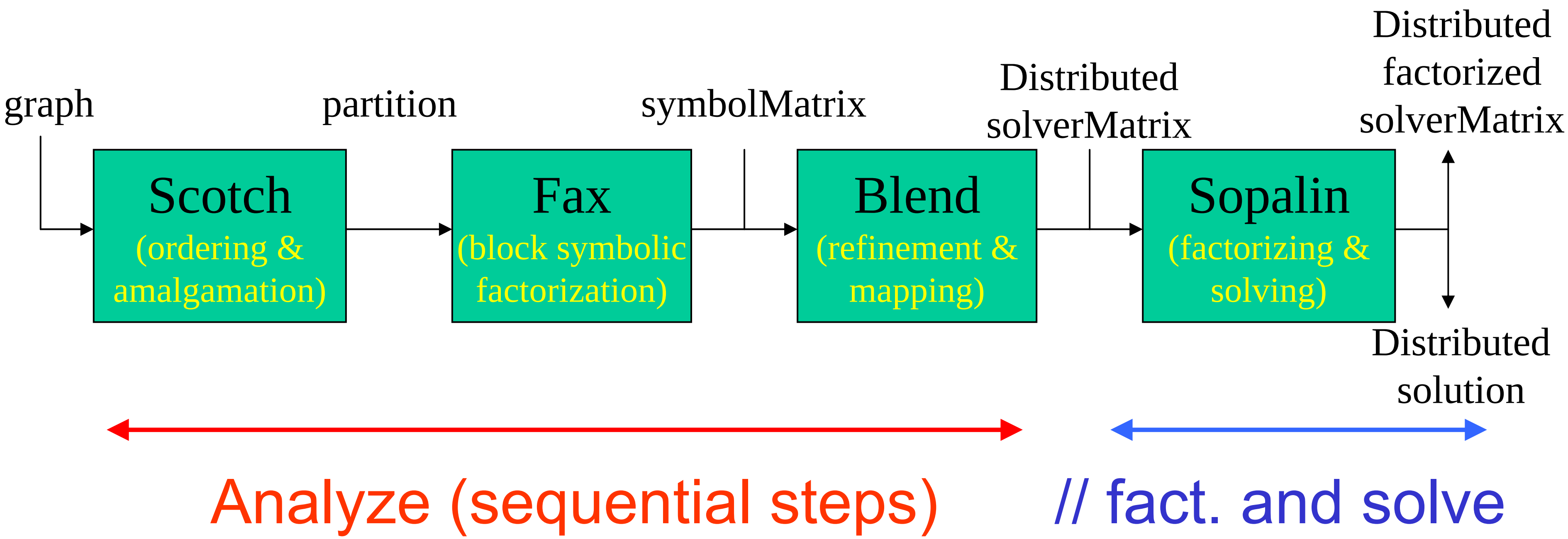
- Goal: we want to adapt a (supernodal) parallel direct solver (PaStiX) to build an incomplete block factorization and benefit from all the features that it provides:

  - Algorithmic is based on linear algebra kernels (BLAS)
  - Load-balancing and task scheduling are based on a fine modeling of computation and communication
  - Modern architecture management (SMP nodes) : hybrid Threads/MPI implementation

Outlines

- Which modifications in the direct solver?
- The symbolic incomplete factorization
- An algorithm to get dense blocks in ILU
- Experiments
- Conclusion

Direct solver chain (in PaStiX)



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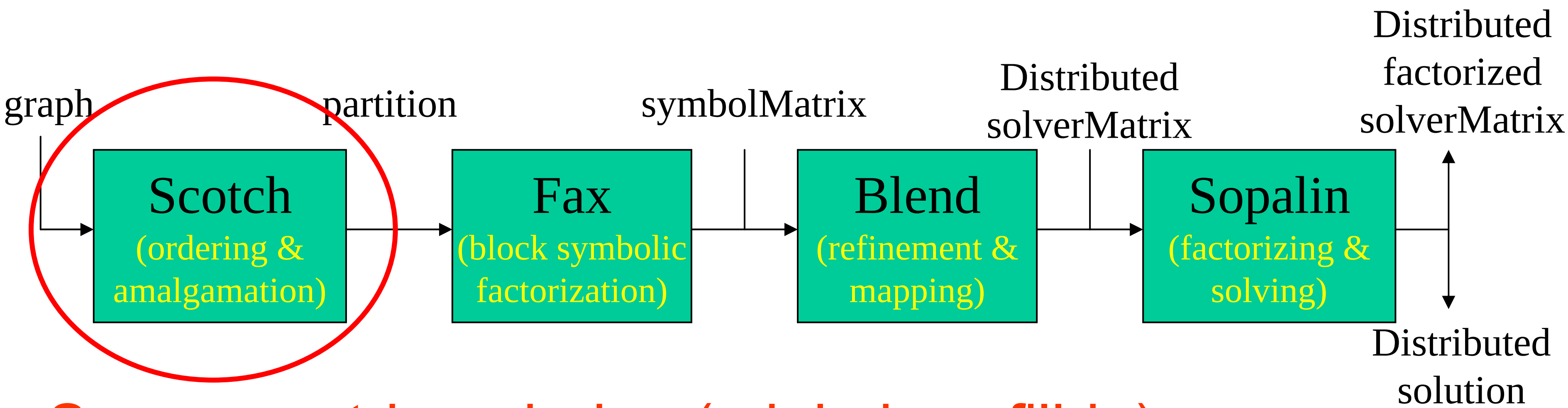

Sparse matrix ordering (minimizes fill-in)

- Scotch: an hybrid algorithm
  - incomplete Nested Dissection
  - the resulting subgraphs being ordered with an Approximate Minimum Degree method under constraints (HAMD)

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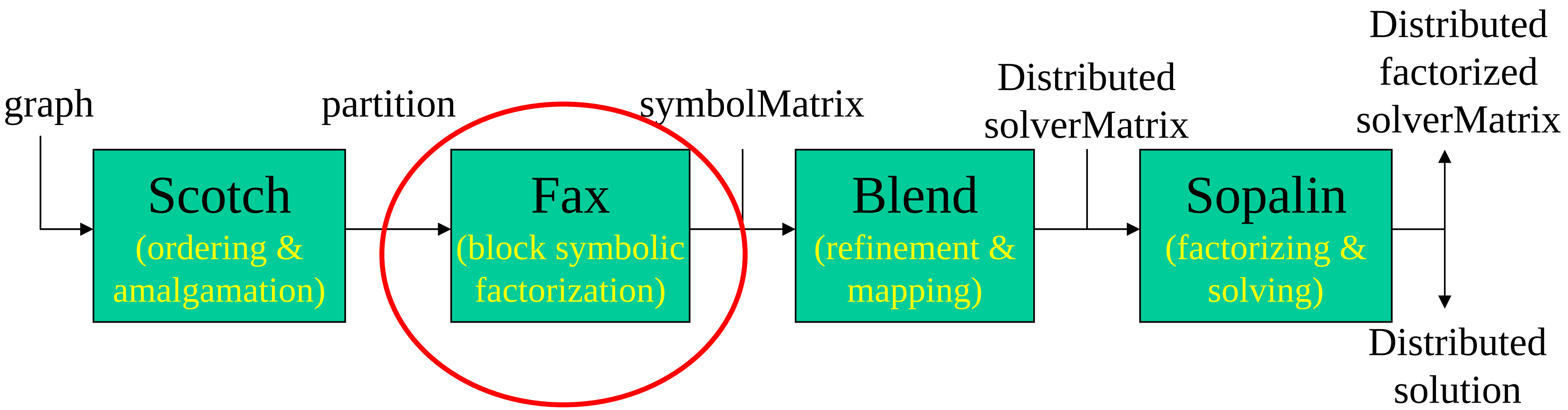
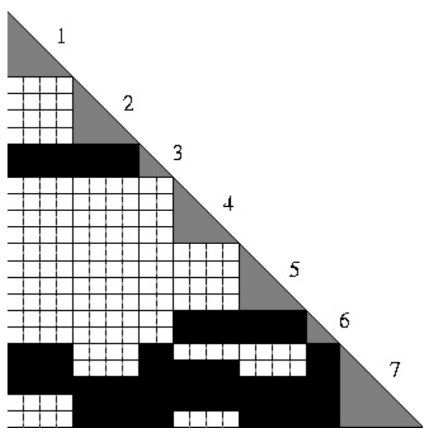


The symbolic block factorization



- ***Q(G,P)→Q(G,P)*=Q(G*,P) => linear in number of blocks!***
- Dense block structures ⇒ only a extra few pointers to store the matrix

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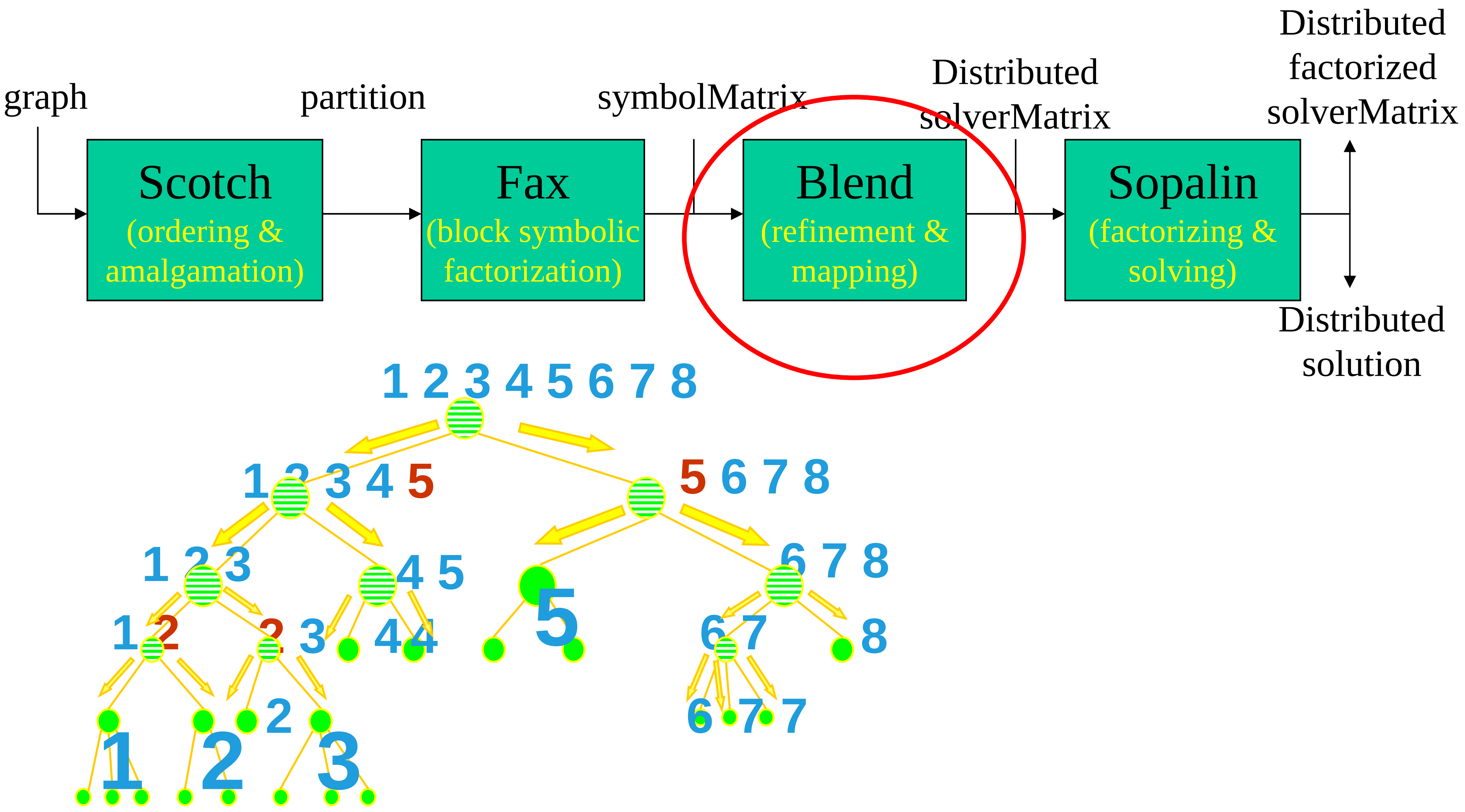

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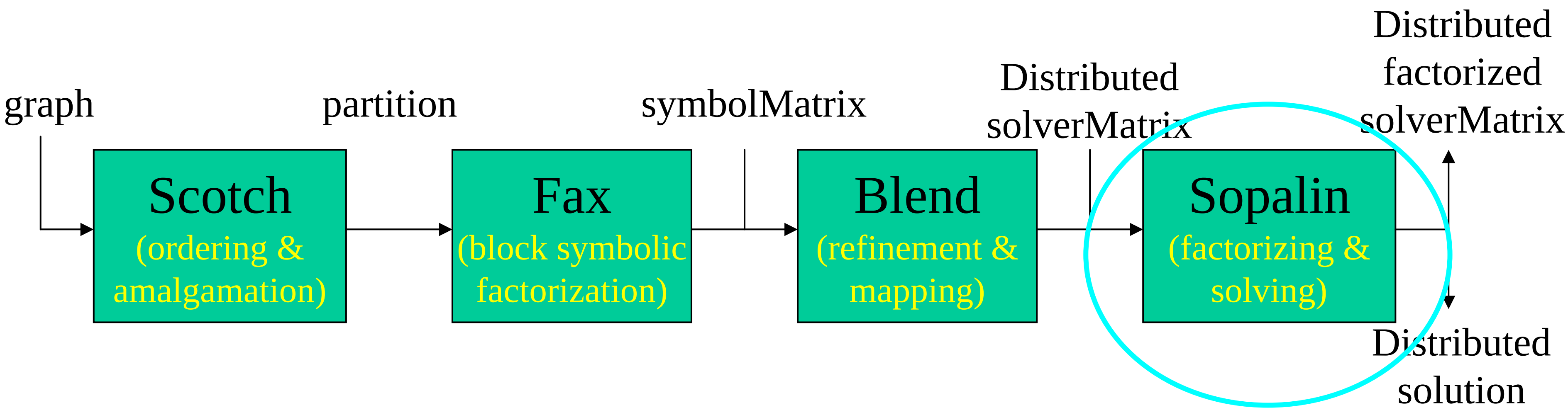

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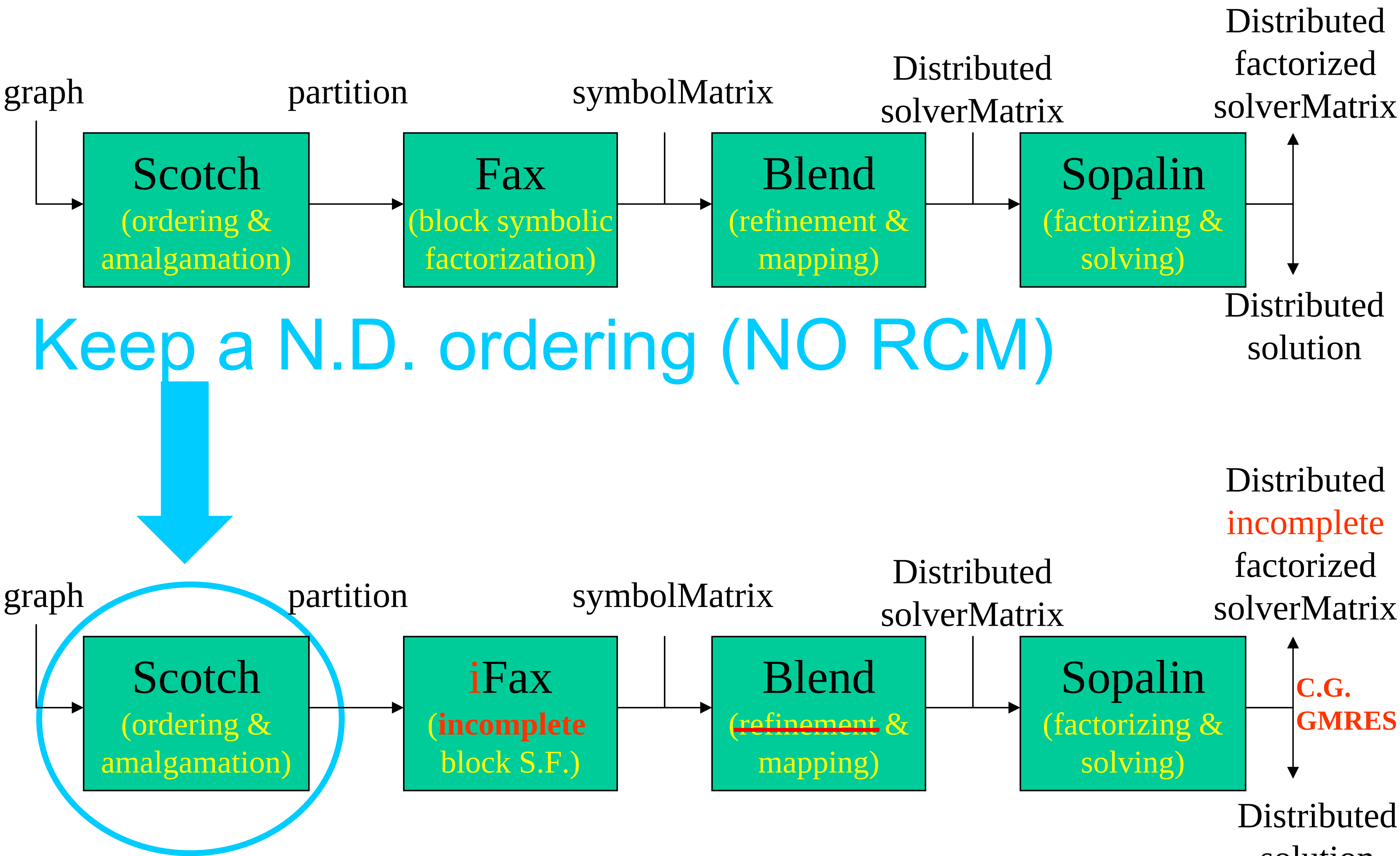
  - Less MPI communication and lower the parallel memory overcost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(ordering & amalgamation)
Fax
(block symbolic factorization)
Blend
(refinement & mapping)
Sopalin
(factorizing & solving)
Distributed solution
Keep a N.D. ordering (NO RCM)
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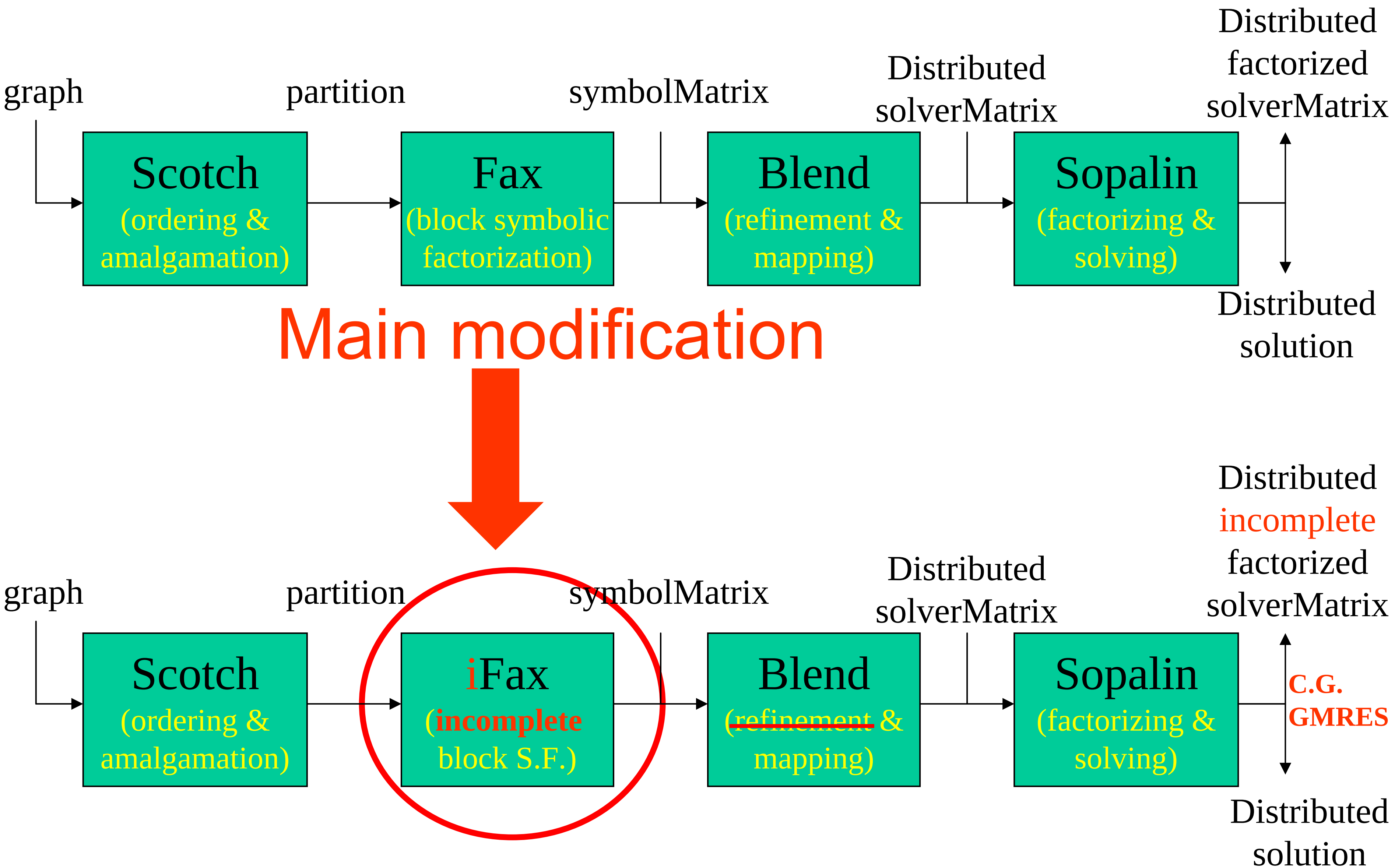
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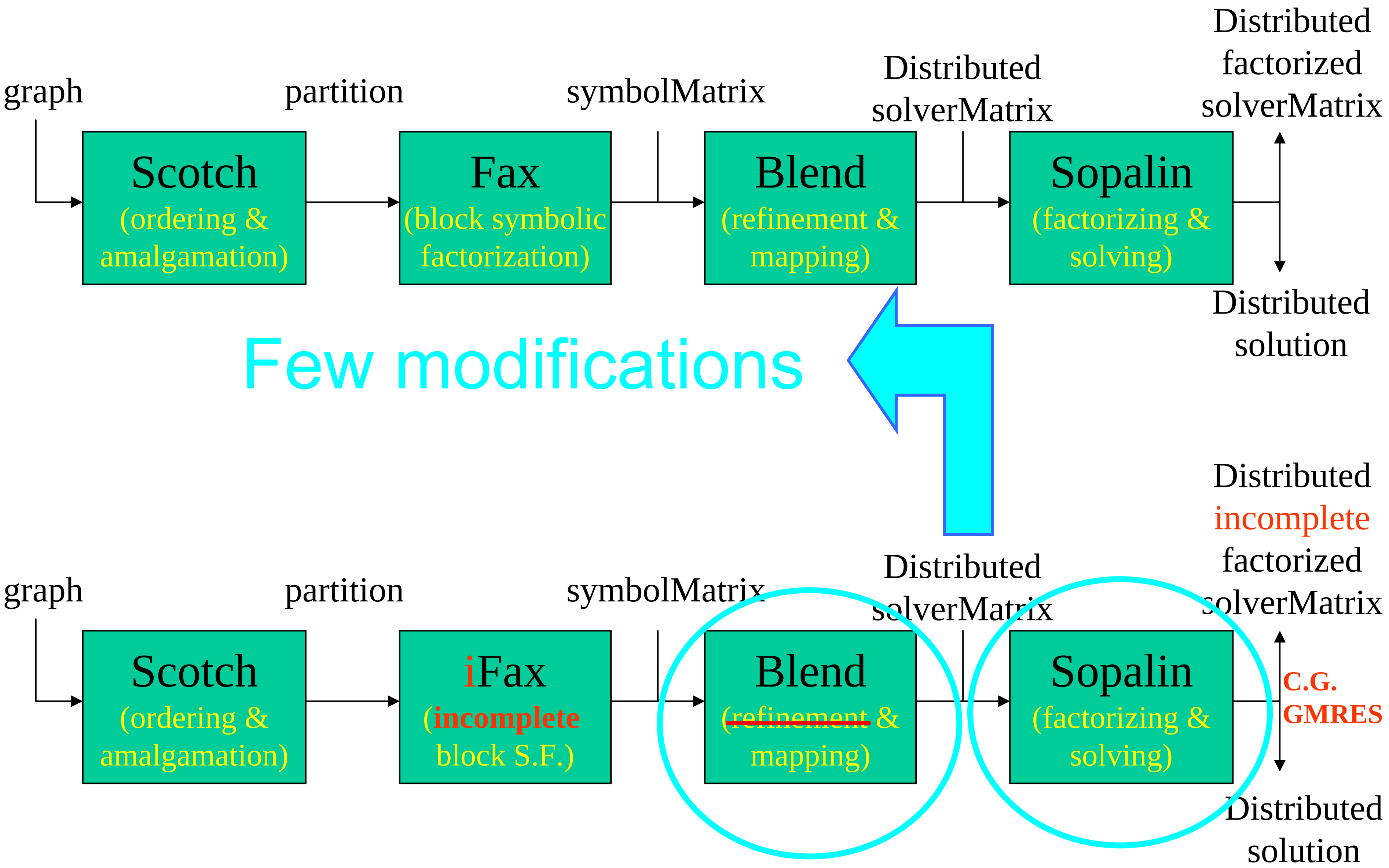
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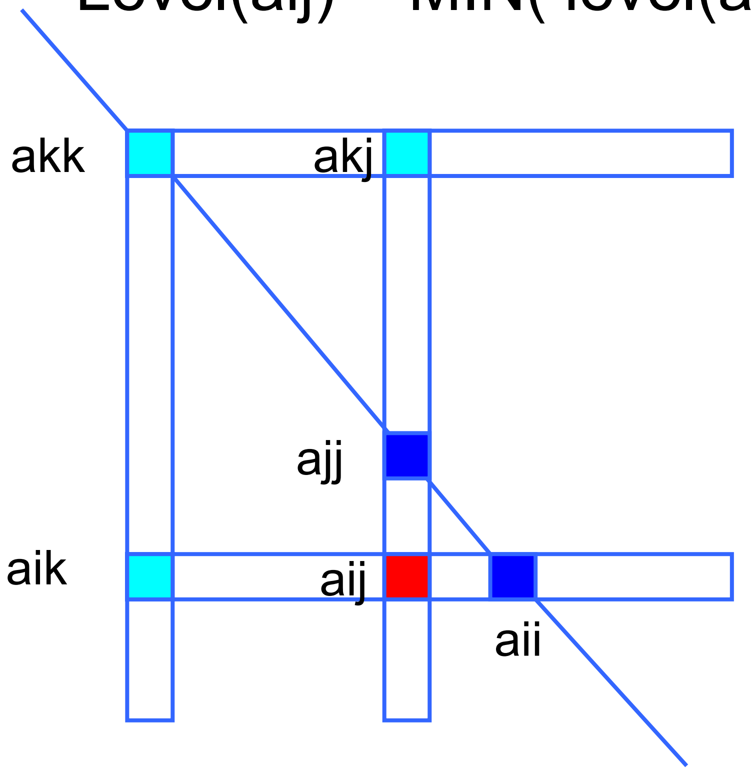
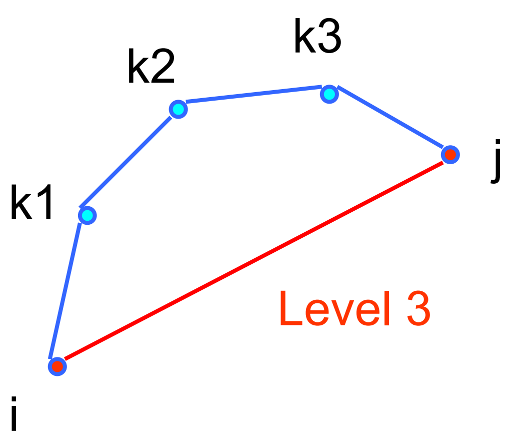
Level based ILU(k)

- Scalar formulation of the level-of-fill:
  Non zero entries of A have a level 0.
  Consider the elimination of the kth unknowns during the fact. then:

  Level(aij) = MIN(level(aij) , level(aik)+level(akj)+1)





k1, k2, k3 < i and j

Level based ILU(k)

- The scalar incomplete factorization have the same asymptotical complexity than the Inc. Fact.

- BUT: it requires much less CPU time

- D. Hysom and A. Pothen gives a practical algorithm that can be easily // (based on the search of elimination^{kj} paths of length <= k+1) [Level Based Incompleted Factorization: Graphs model and Algorithm (2002)]

Level based ILU(k)

- In a FEM method a mesh node corresponds to several Degrees Of Freedom (DOF) and in this case we can use the node graph instead of the adj. graph of A i.e. :

  $Q(G,P) \rightarrow Q(G,P)^k = Q(G^k,P)$ P = *partition of mesh nodes*

- This means the symbolic factorization will have a complexity in the respect of the number of nodes whereas the factorization has a complexity in respect to the number of DOF.

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How to build a dense block structure in ILU(k) factors ?

- First step: find the exact supernode partition in the ILU(k) NNZ pattern

- In most cases, this partition is too refined (dense blocks are usually too small for BLAS3)

- Idea: we allow some extra fill-in in the symbolic factor to build a better block partition

➢ Ex: How can we make bigger dense blocks if we allow 20% more fill-in ?

How to build a dense block structure in ILU(k) factors ?

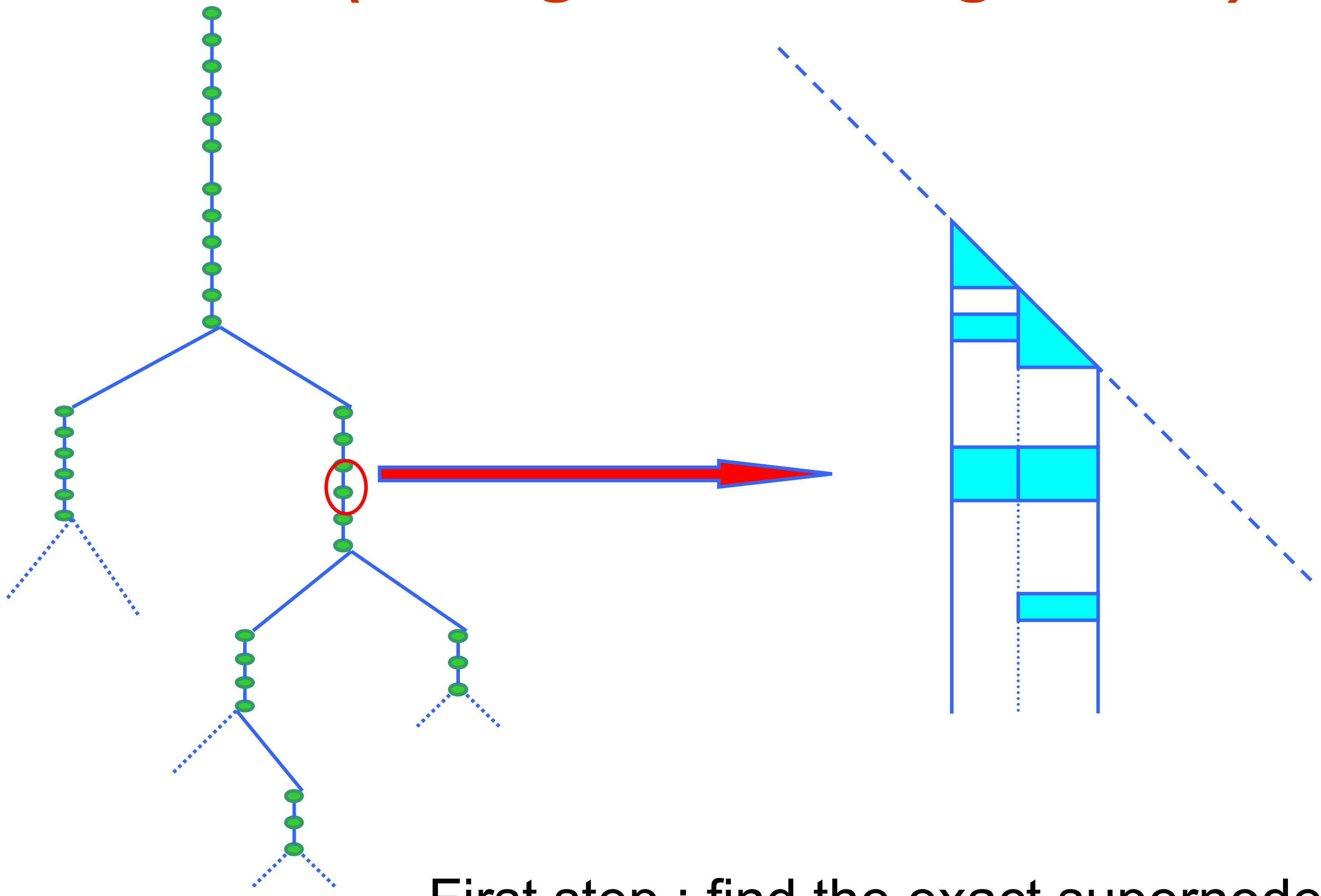
- We imposed some constraints:

➢ any permutation that groups columns with similar NNZ pattern should not affect G^k

➢ any permutation should not destroy the elimination tree structure

=> We impose the rule « merge only with your father… » for the supernode

Finding an approximated Supernodes Partition (amalgamation algorithm)



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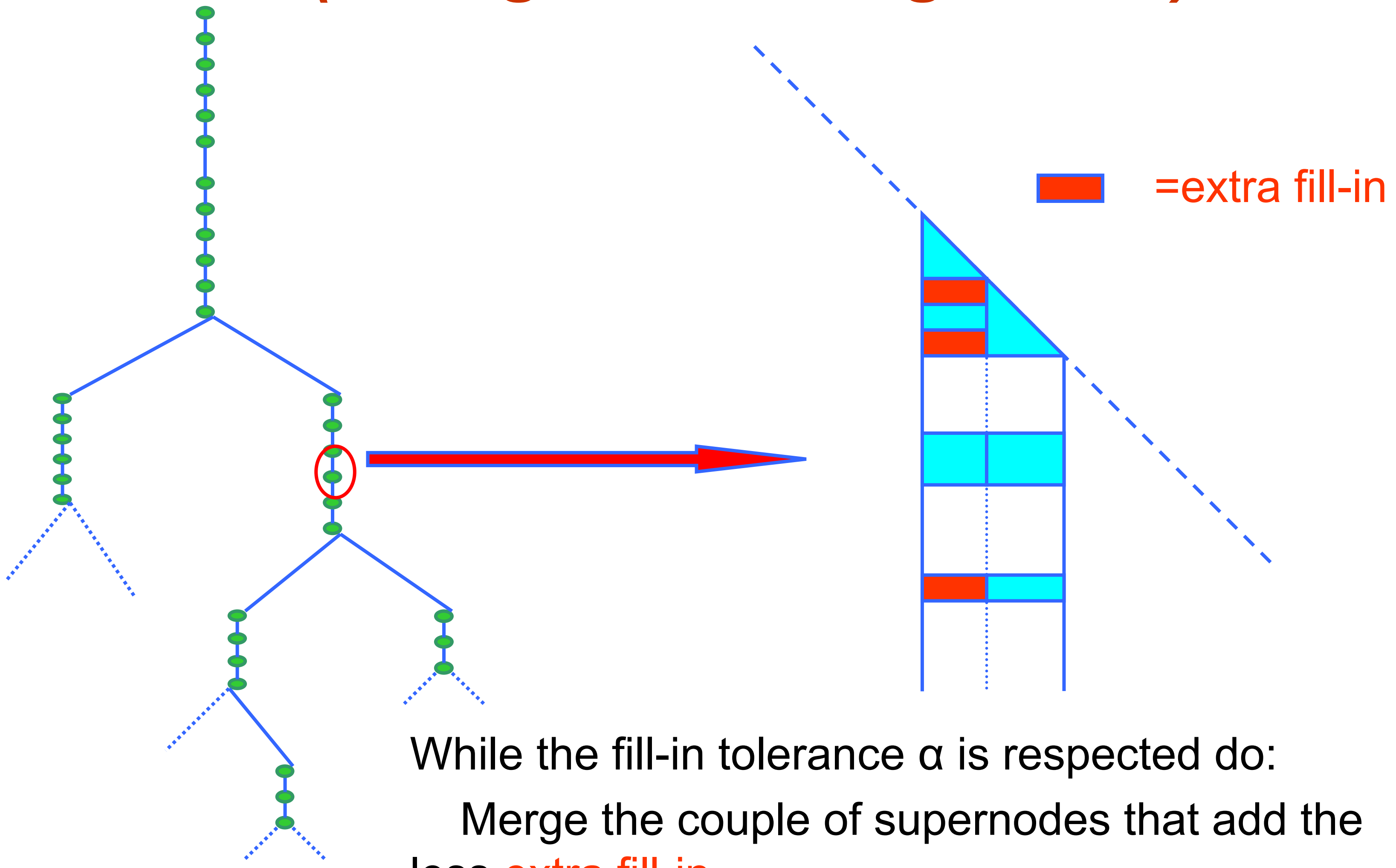

While the fill-in tolerance α is respected do:

Merge the couple of supernodes that add the less extra fill-in

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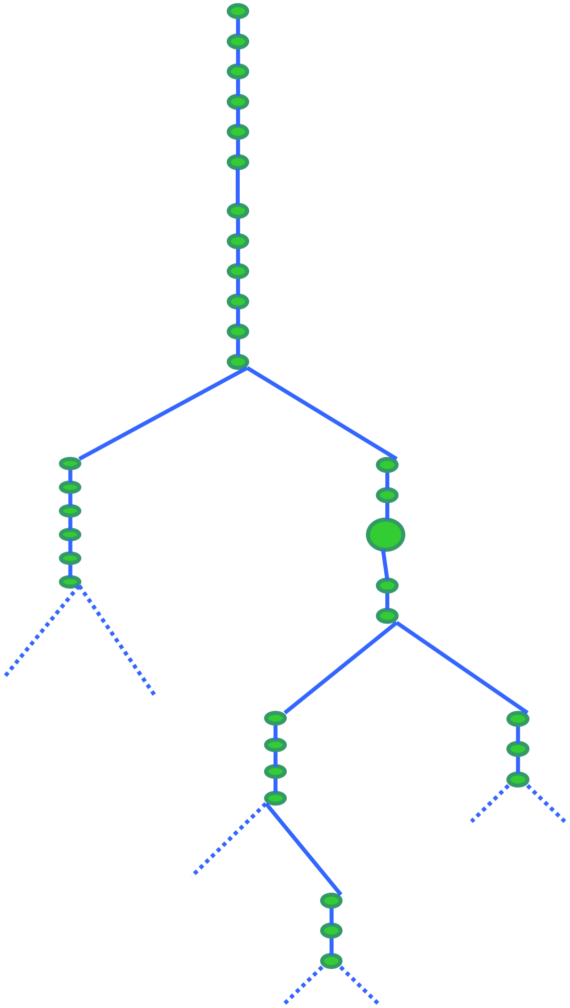

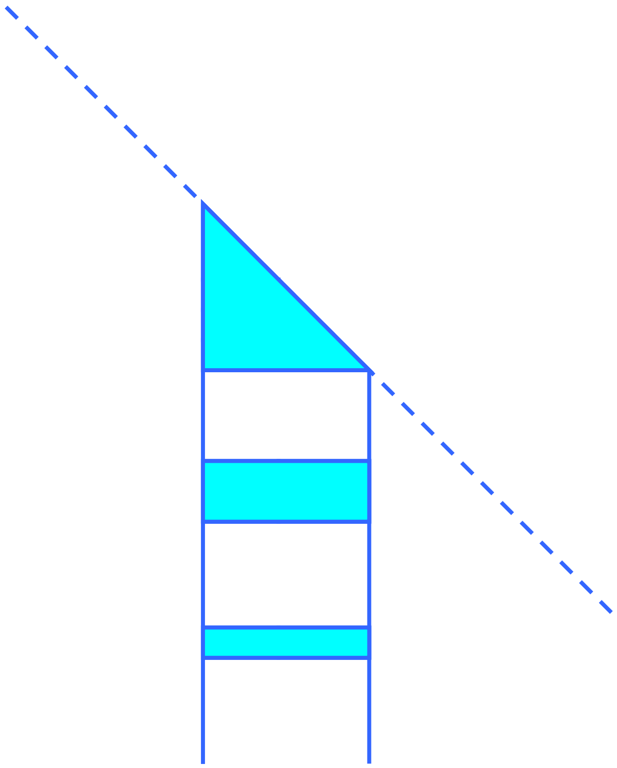

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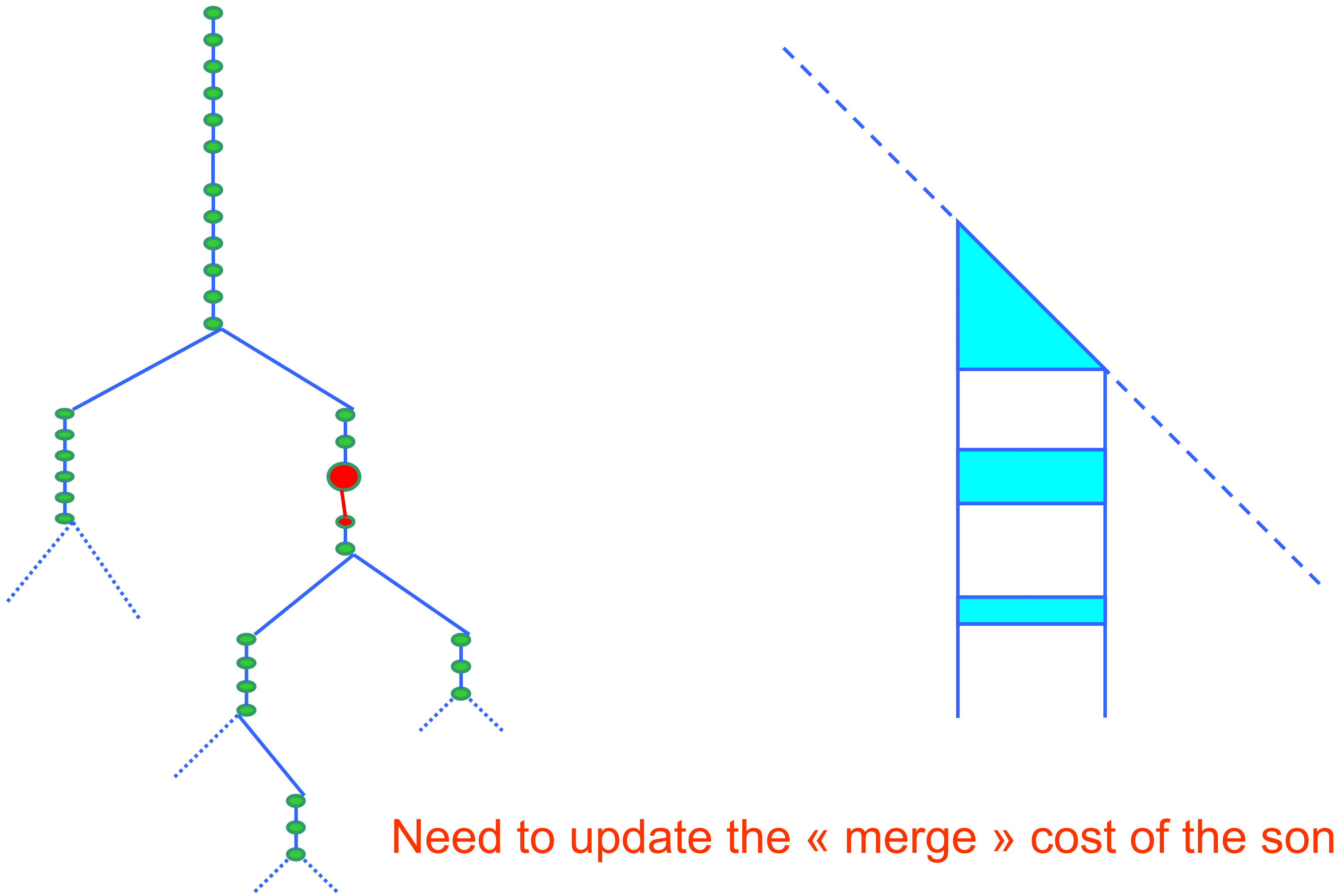

Need to update the « merge » cost of the son

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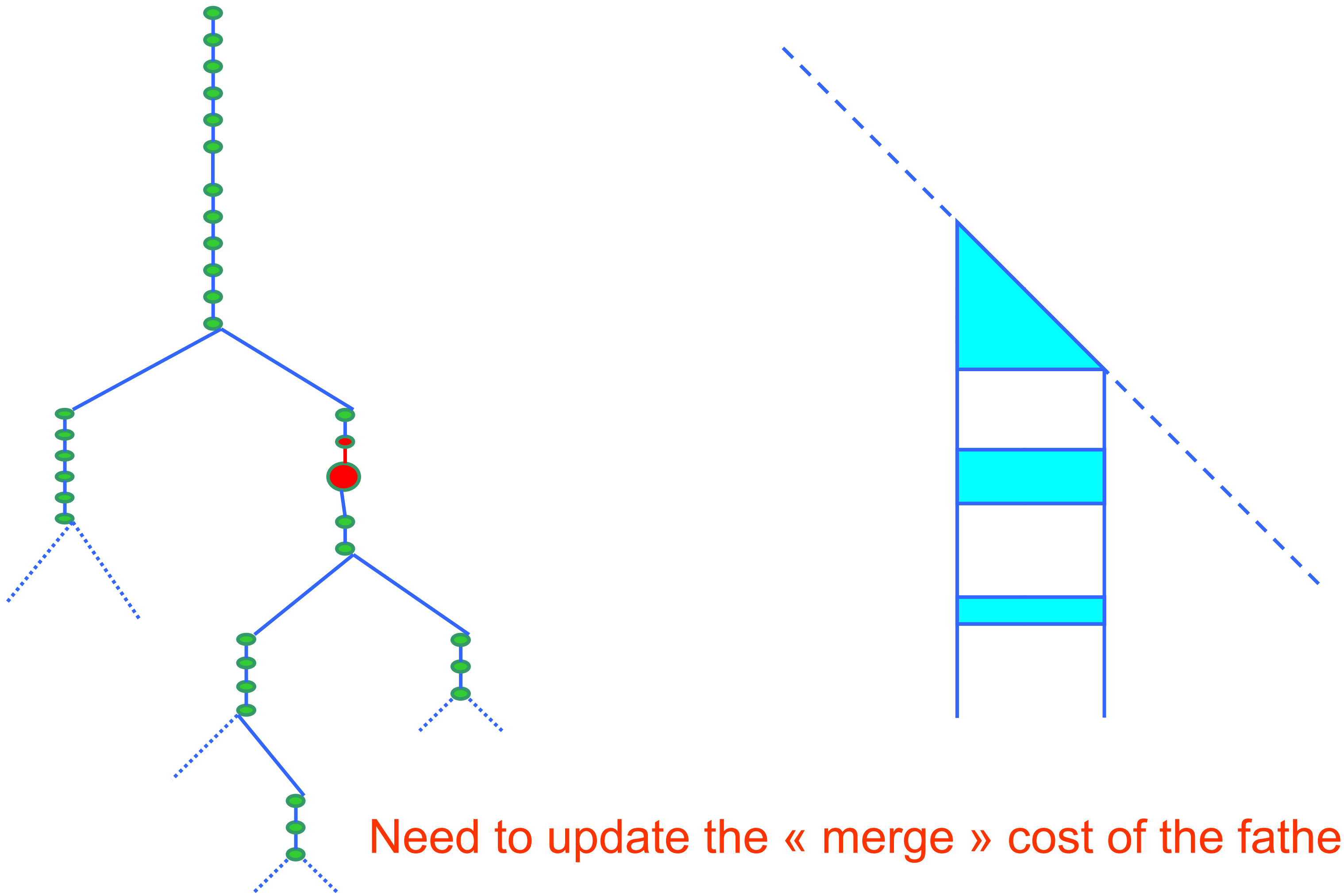

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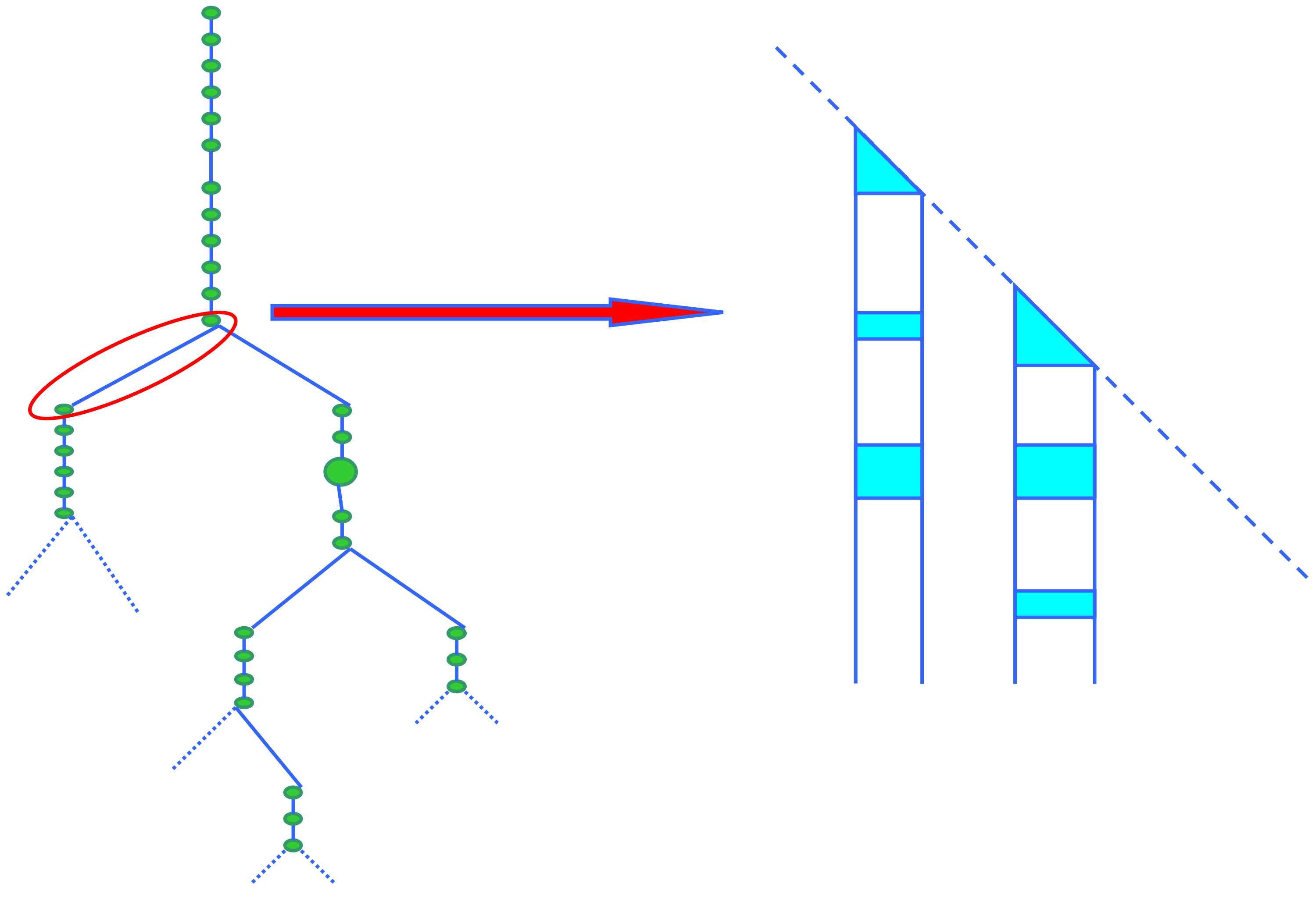

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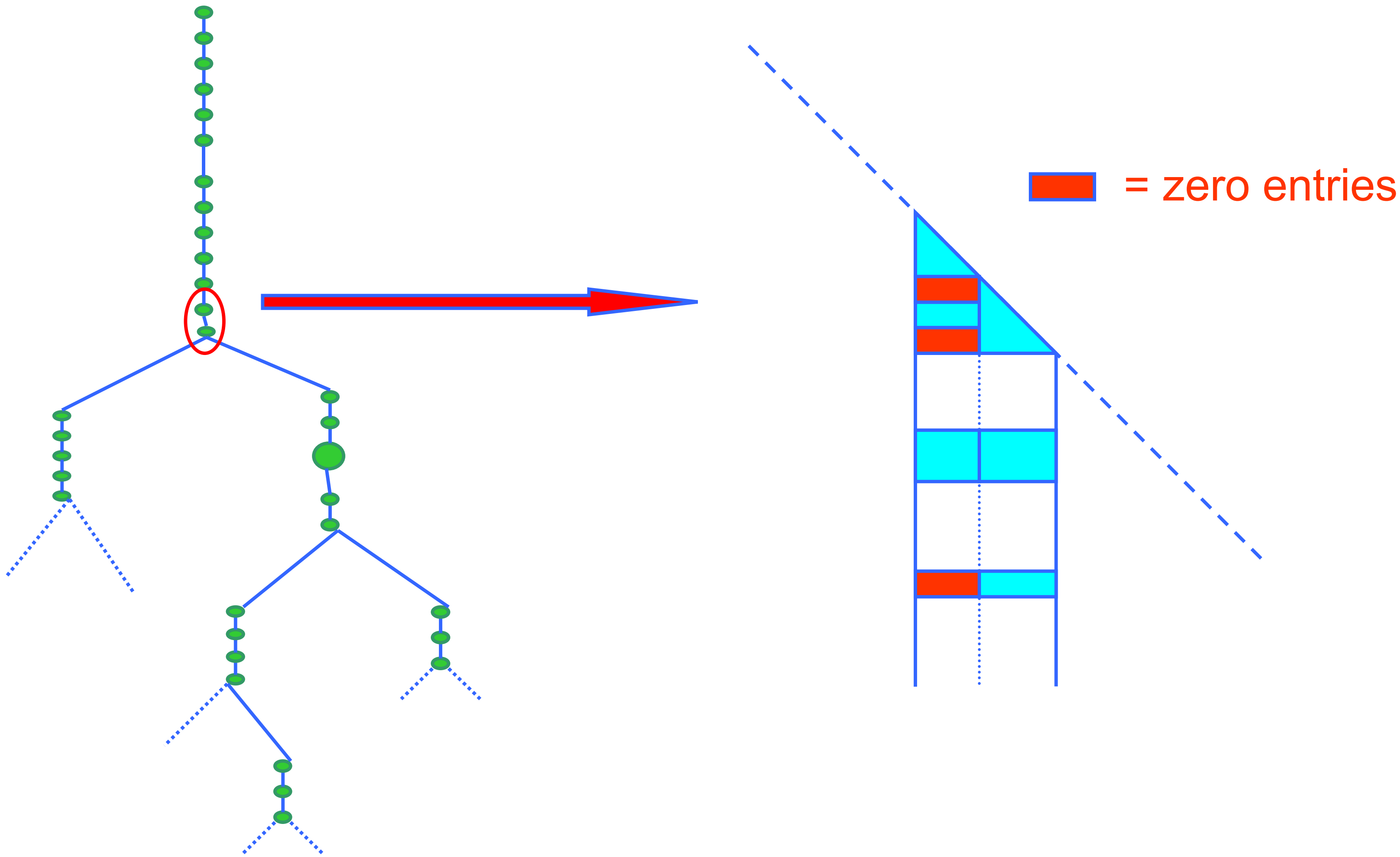

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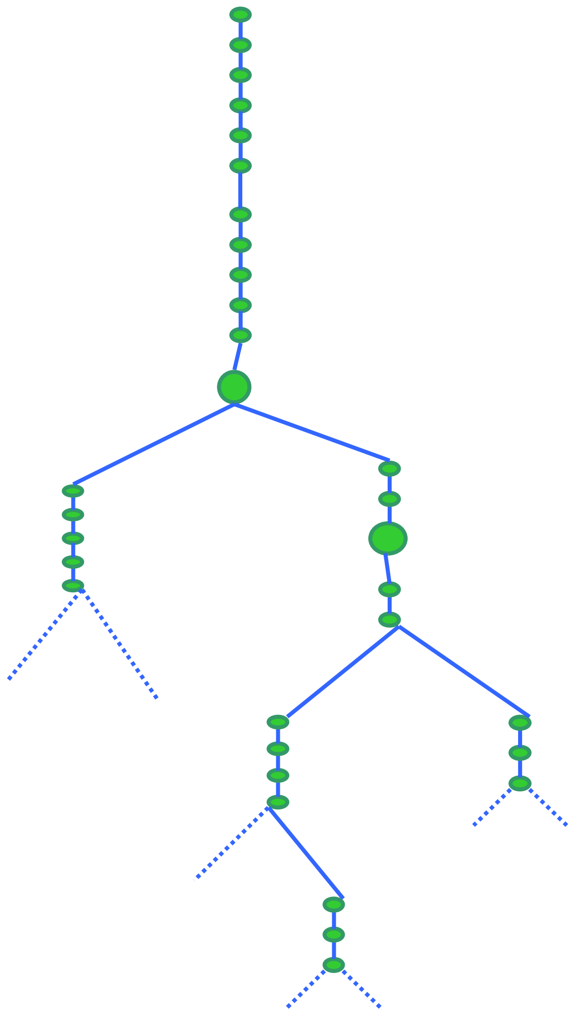

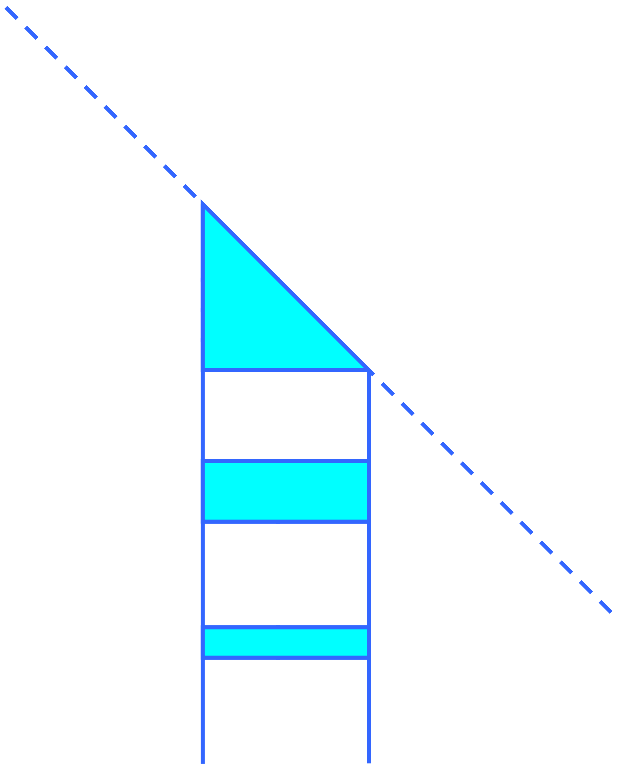

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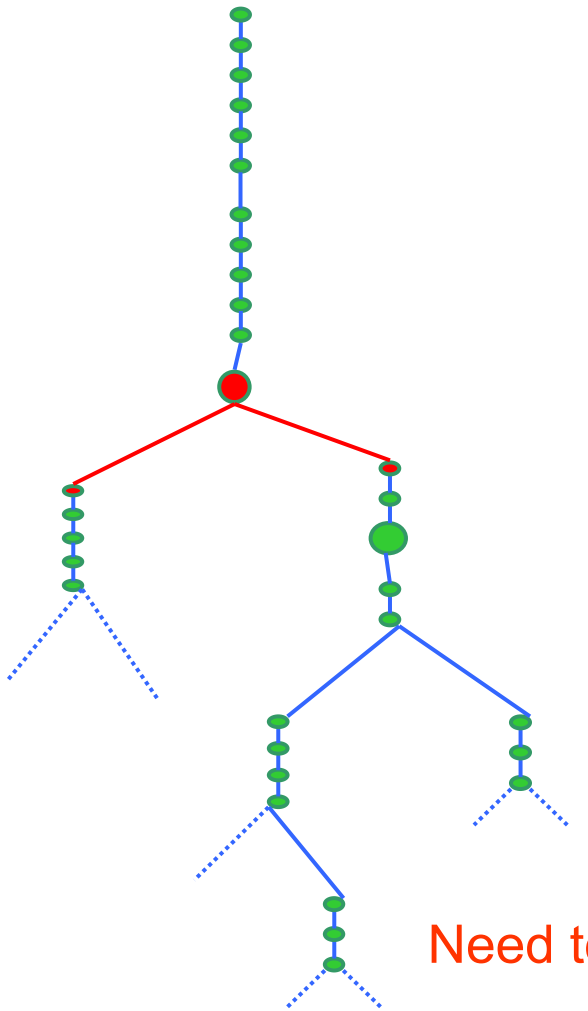

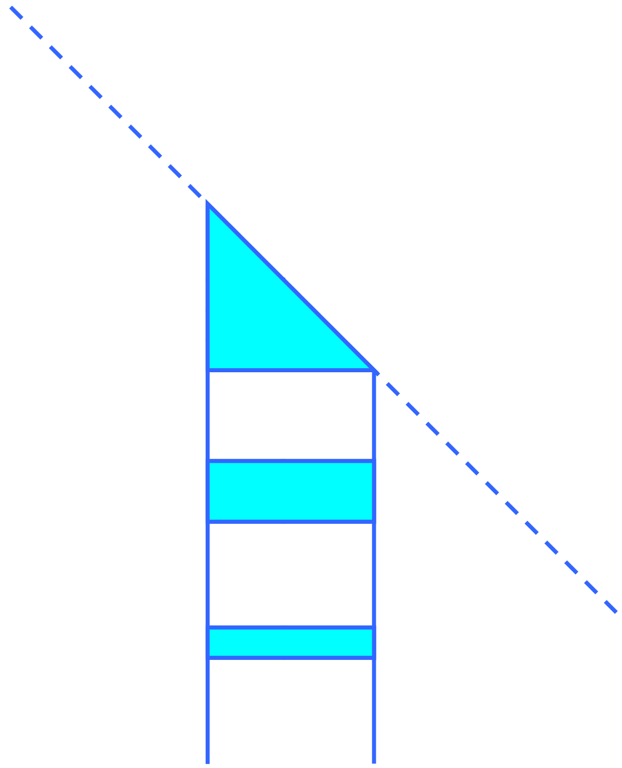

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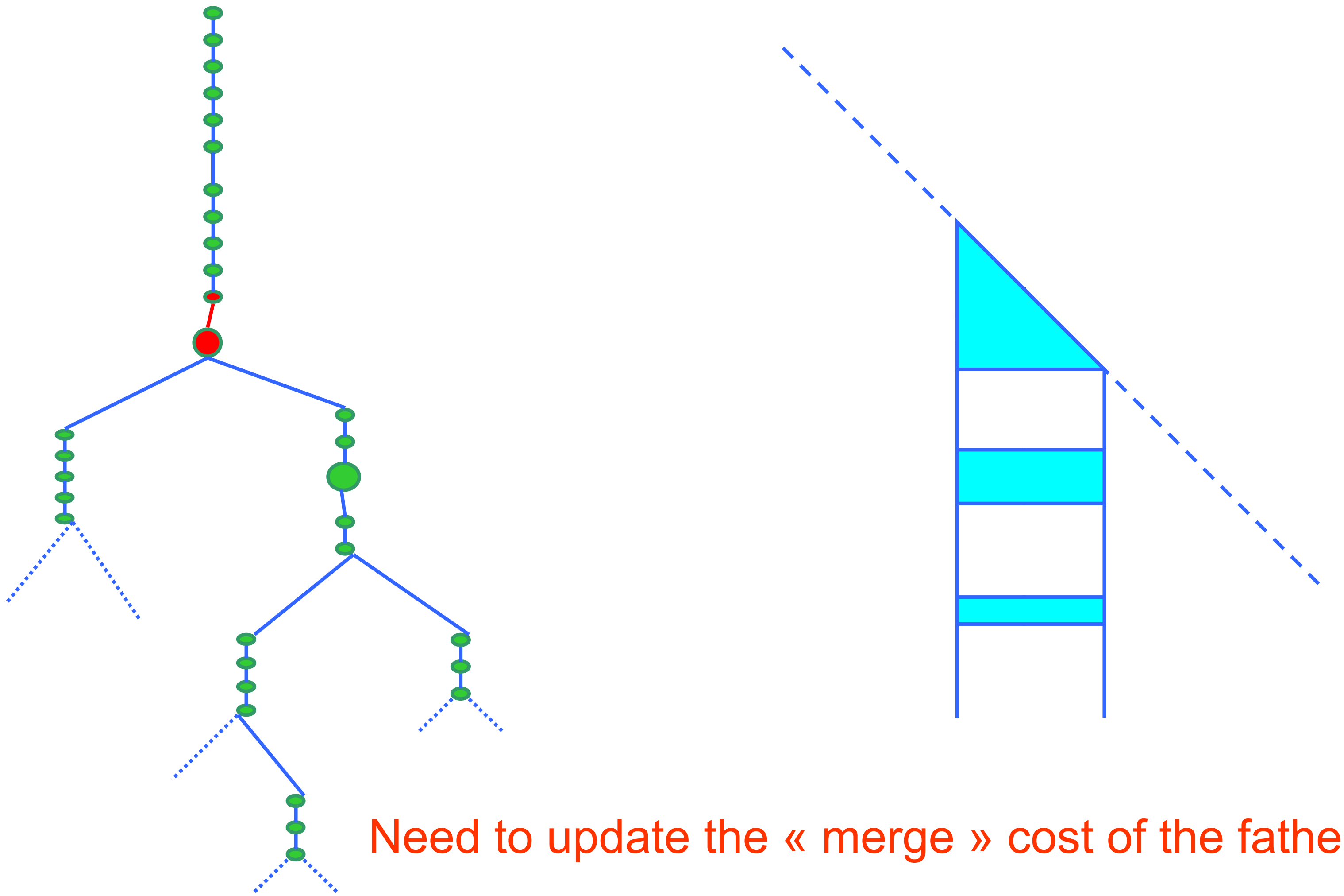

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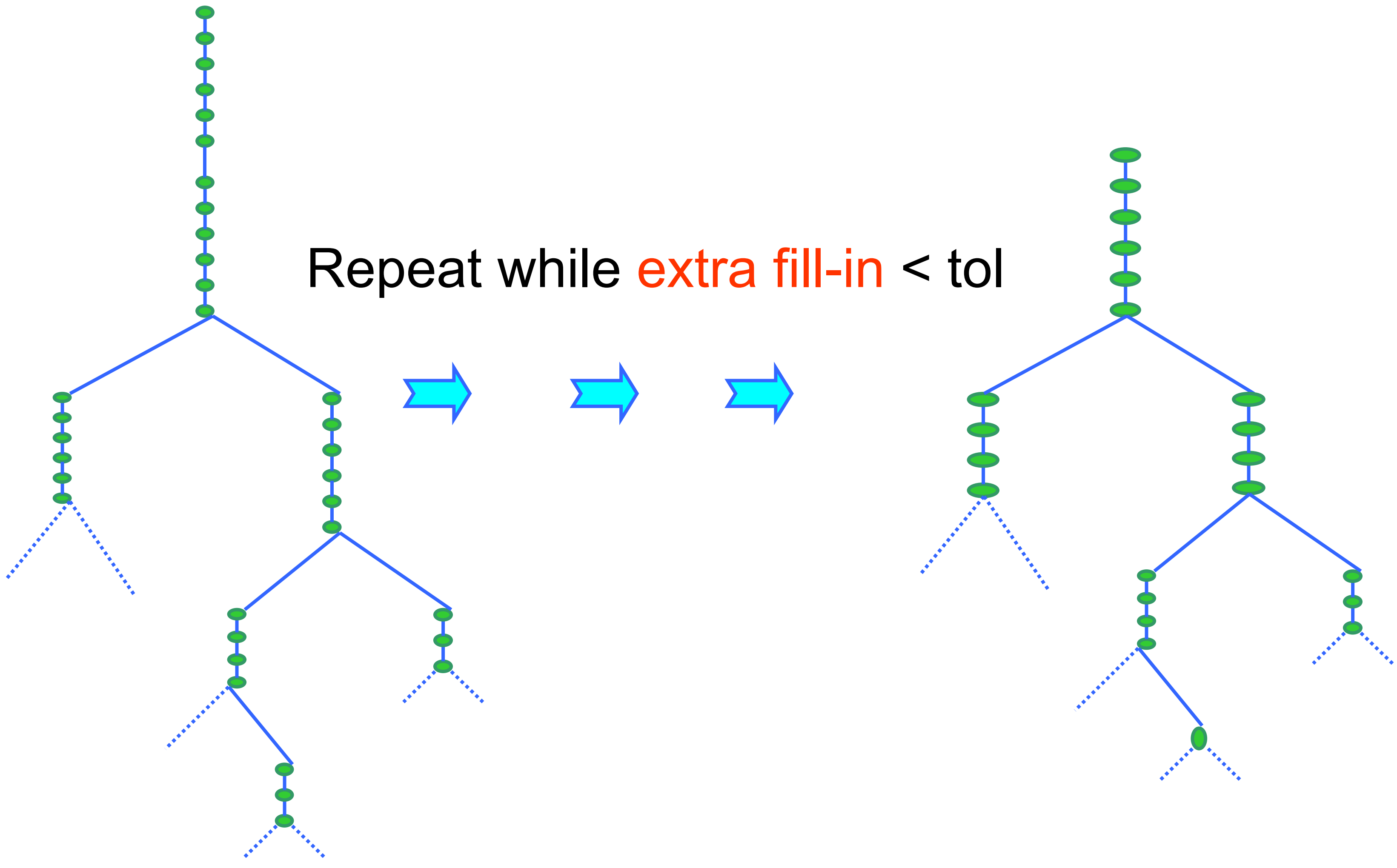


Cost of the algorithm

- The approximate supernode merging algorithm is really cheap compare to the other steps

- At each step: recompute fill-add for modified (son-father) couples and maintain the heap sort.

- Complexity bound by $O(D.N_0 + N_0.Log(N_0))$
  N_0 : number of exact supernodes in ILU factors
  D : maximum number of extradiagonal blocks in a block-column

Numerical experiments

- Results on IBM power5 + Switch “Federation”
- All computations were performed in double precision
- Iterative accelerator was GMRES (no restart)
- Stopping criterion for iterative accelerators was a relative residual norm (||b-A.x||/||b||) of 1e-7

Test cases:

- AUDIKW_1 : Symmetric matrix (Parasol collection)
  n = 943,695 nnz(A) = 39,297,771
  With direct solver : nnz(L) = 31 x nnz(A)
  total solution in 115s on 16 procs
  ➔ 3D

- SHIPSEC5 : Symmetric matrix (Parasol collection)
  n = 179,860 nnz(A) = 4,966,618
  With direct solver : nnz(L) = 11 x nnz(A)
  total solution in 7s on 16 procs
  ➔2D

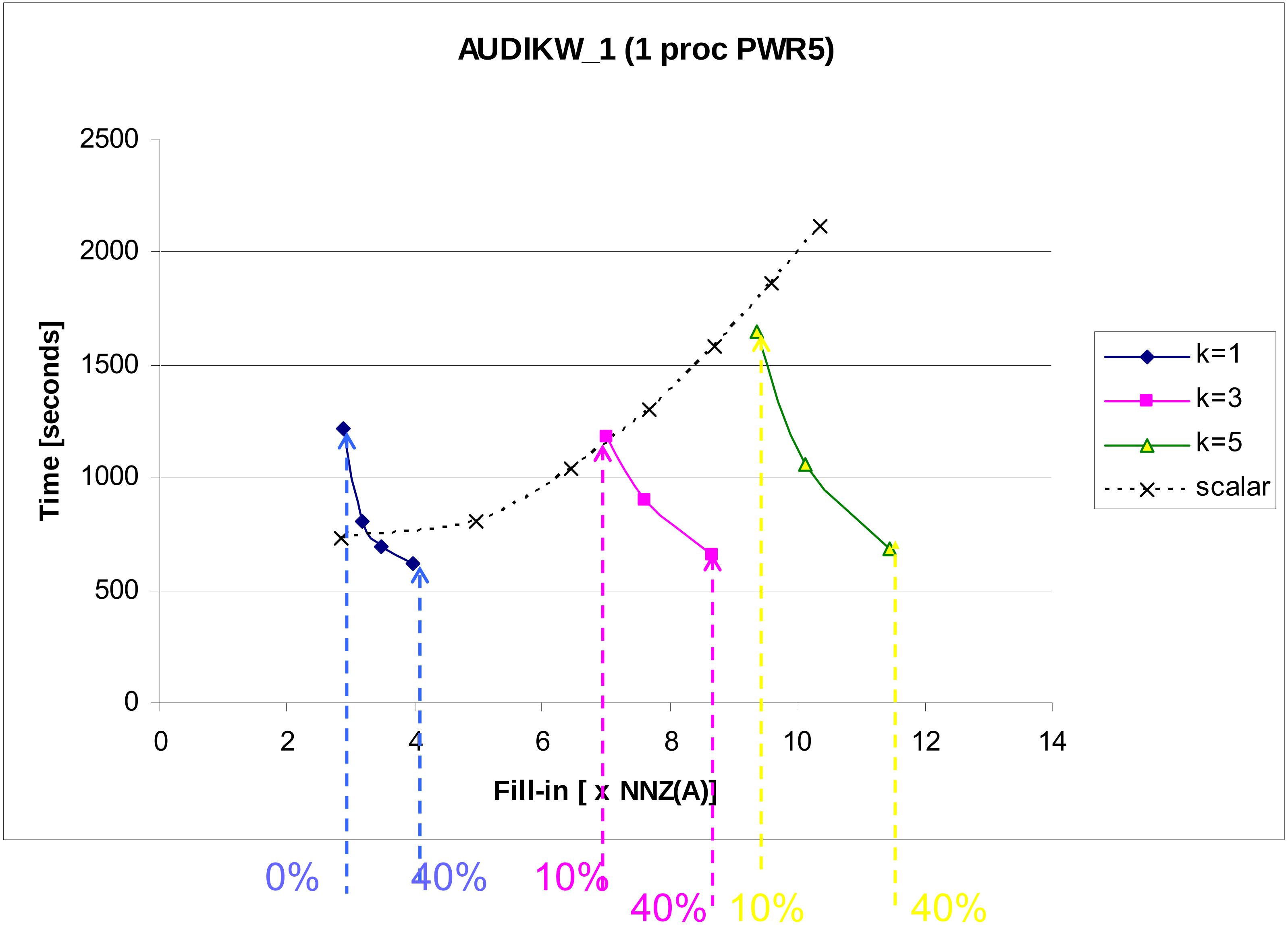
Effect of amalgamation ratio α

AUDIKW_1: n = 943,695 nnzA=39,297,771

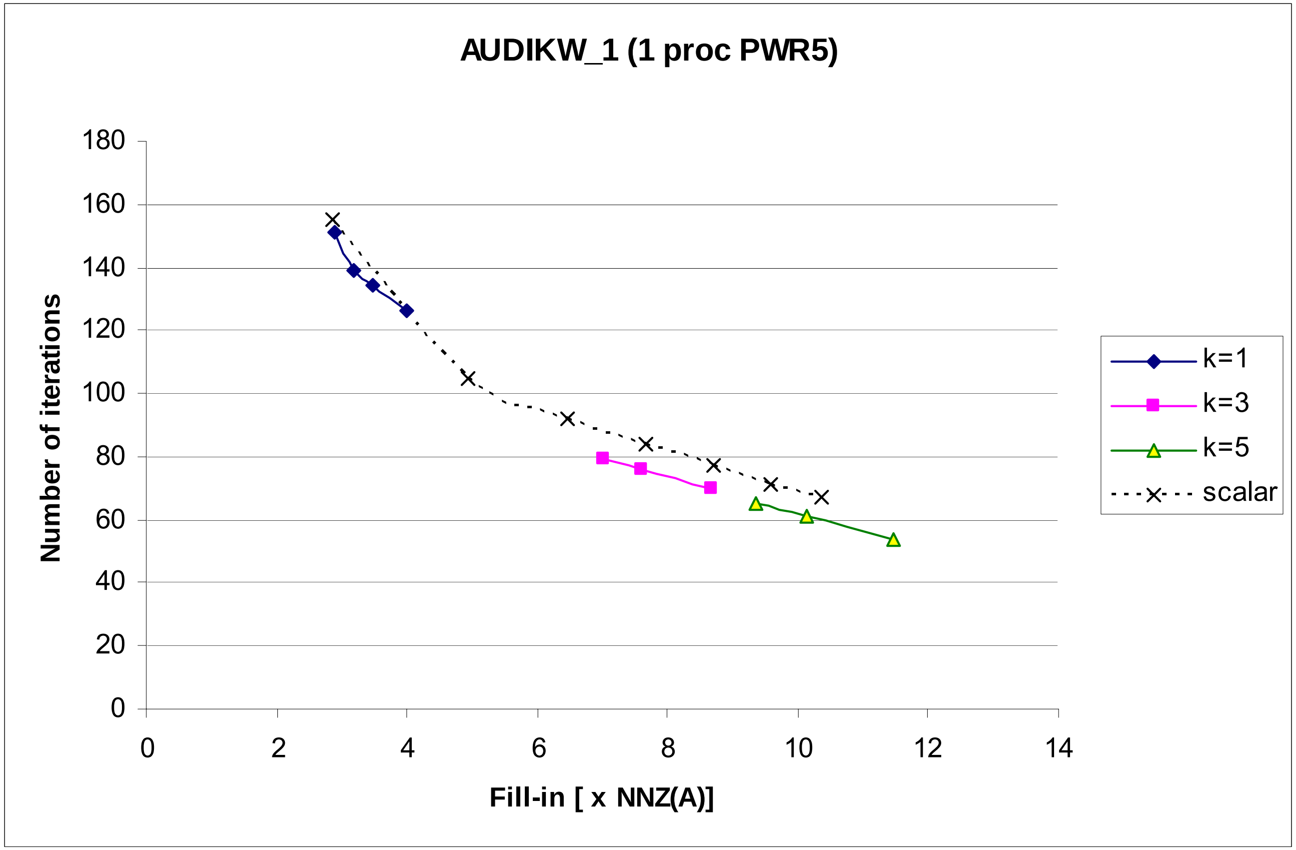
K	α	CBLK	BLOCKS	Amalg	Fact.	Tr. solve
1	0%	300,386	11,893,366	4.74	167.19	6.94
1	20 %	133,102	4,422,368	8.18	71.72	4.67
1	40 %	83,168	2,564,865	9.59	53.10	4.50
3	0%	292,096	27,099,992	8.63		
3	20 %	85,759	6,255,623	14.18	293.33	7.96
3	40 %	41,515	2,278,474	15.71	163.88	7.00
5	0%	275,012	35,399,482	11.04		
5	20 %	62,203	6,453,393	17.23	518.57	8.86
5	40 %	27,915	1,890,939	19.00	258.11	7.80

Sequential Time: total fact. + solve.

Res. precision= 1e-7



Number of Iterations



Parallel Time: AUDIKW_1

		1 processor			16 processors		
K	α	Fact	TR solv	Total	Fact	TR Solv	Total
1	20 %	74.5	4.59	690.1	21.4	0.51	91.5
1	40 %	56.4	4.44	620.3	12.7	0.42	67.0
3	20 %	331.1	7.97	936.8	39.2	0.91	108.7
3	40 %	194.6	7.57	732.0	18.6	0.66	65.7
5	20 %	518.5	8.86	1058.9	52.3	1.16	123.1
5	40 %	258.1	7.80	679.3	21.2	0.78	63.3

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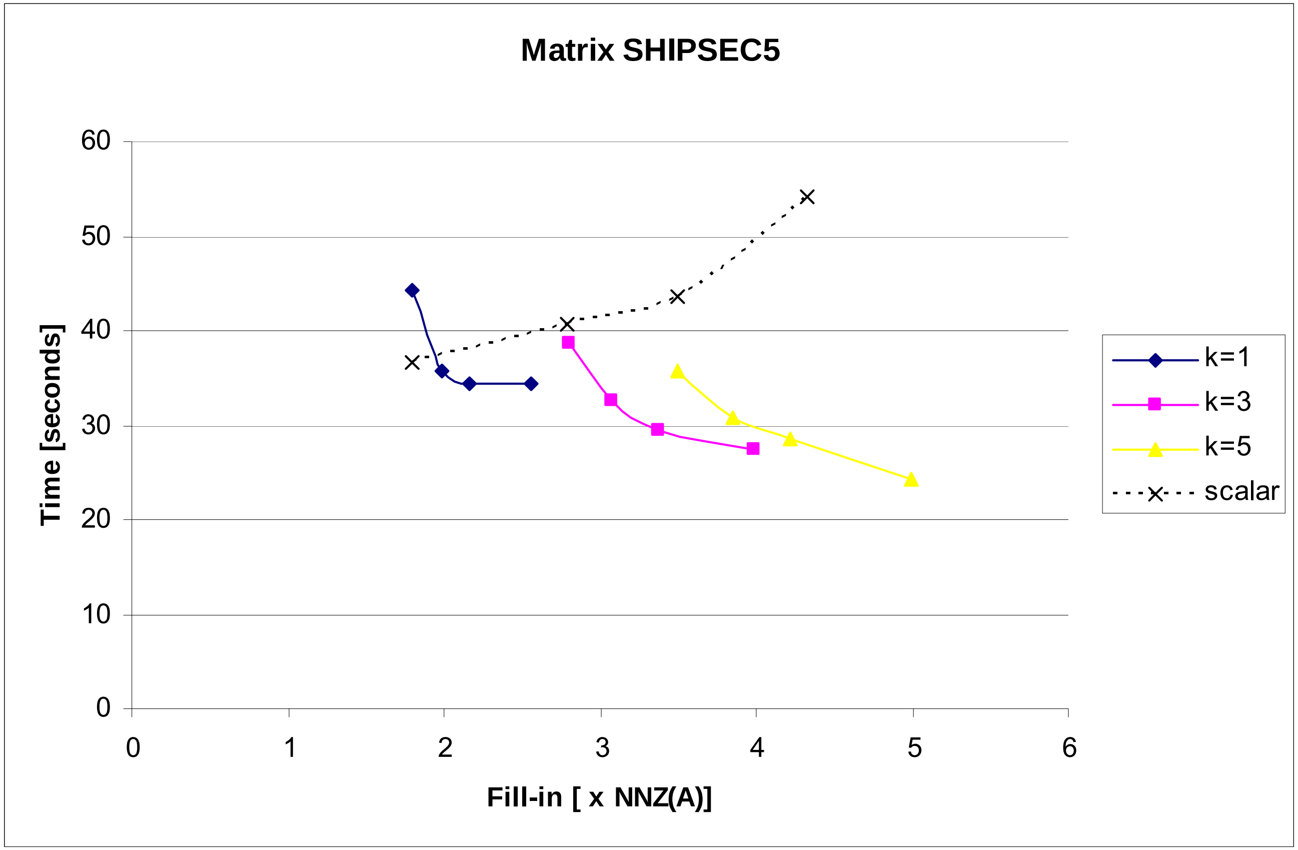

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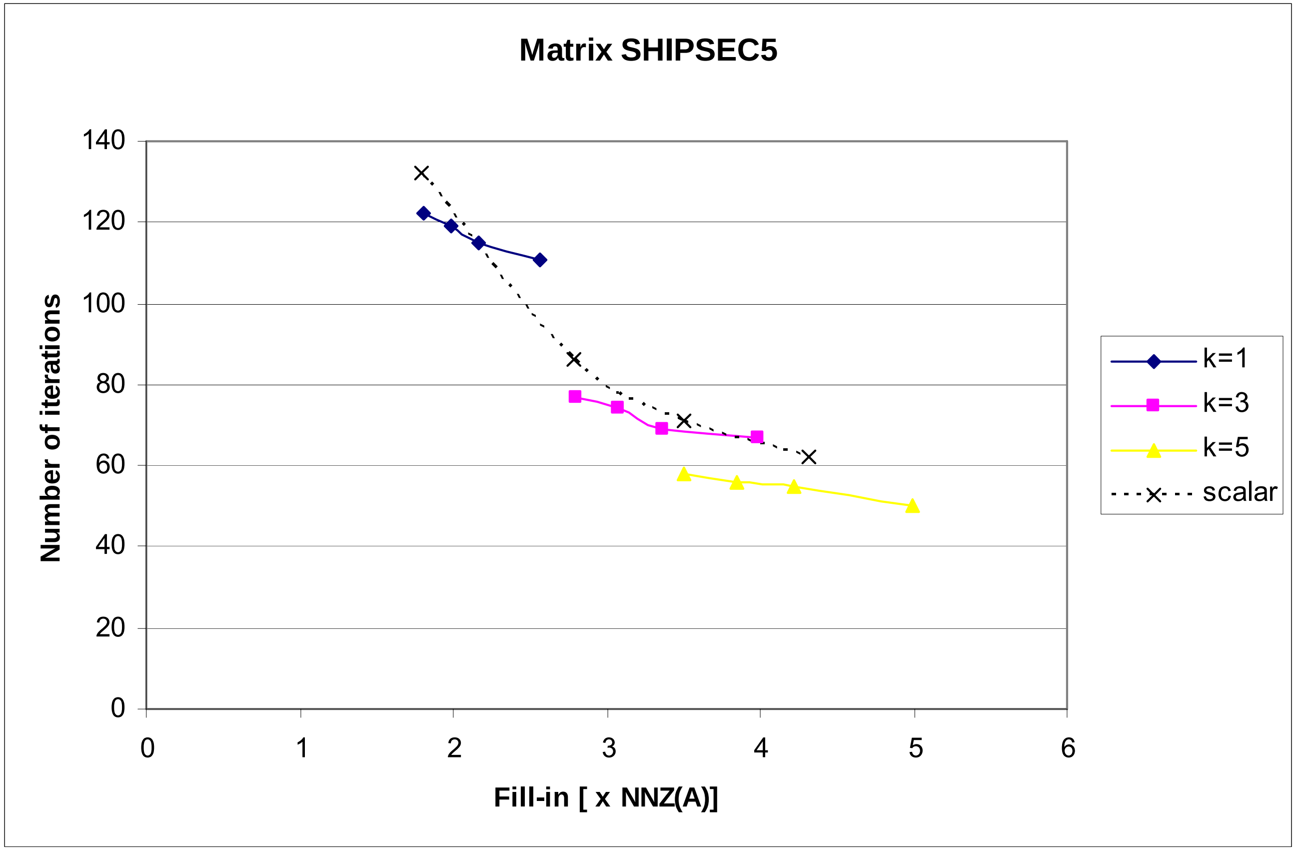

Conclusion

- ⇒This method provides an efficient parallel implementation of ILU(k) precon. (and does not depends on the numbers of proc.)
- ⇒The amalg. algorithm could be improved by relaxing the constraint of the « merge only with your father » but this requires further modifications in the solver chain.

Symbolic ILU(k) Audikw_1

Level of fill	Symb. Facto.	Num. Fact.
K=1	16.8	74.97
K=3	73.8	466.94
K=5	131.11	1010.4

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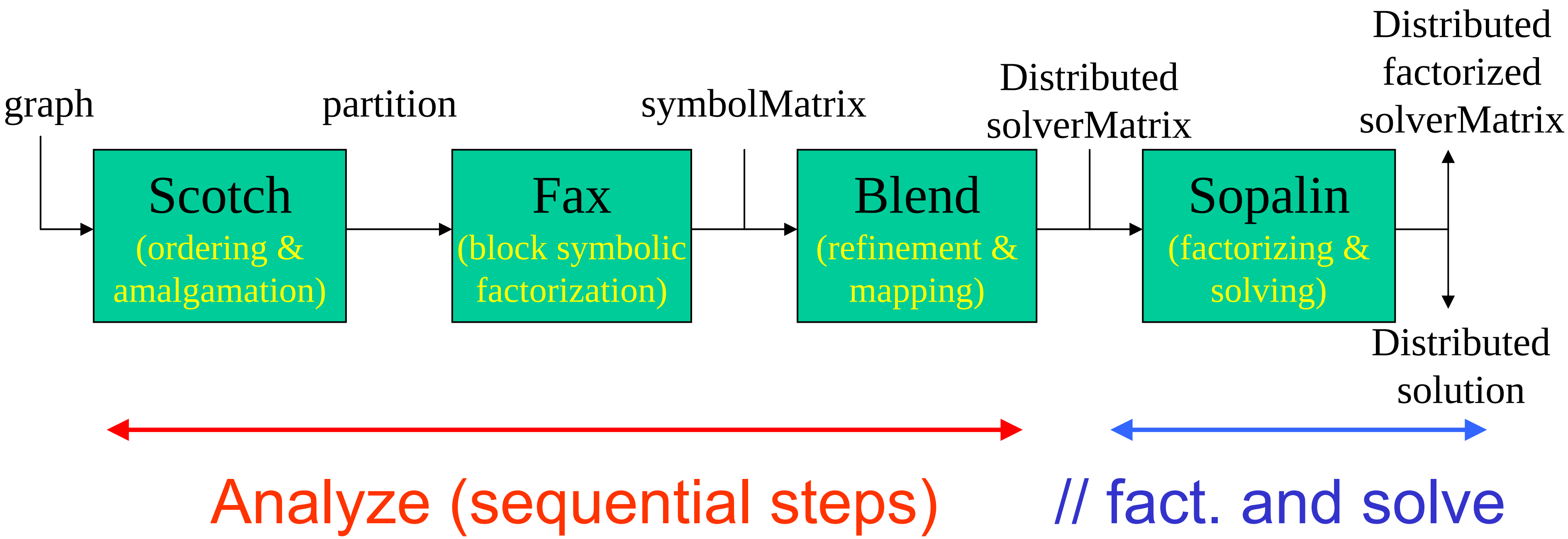

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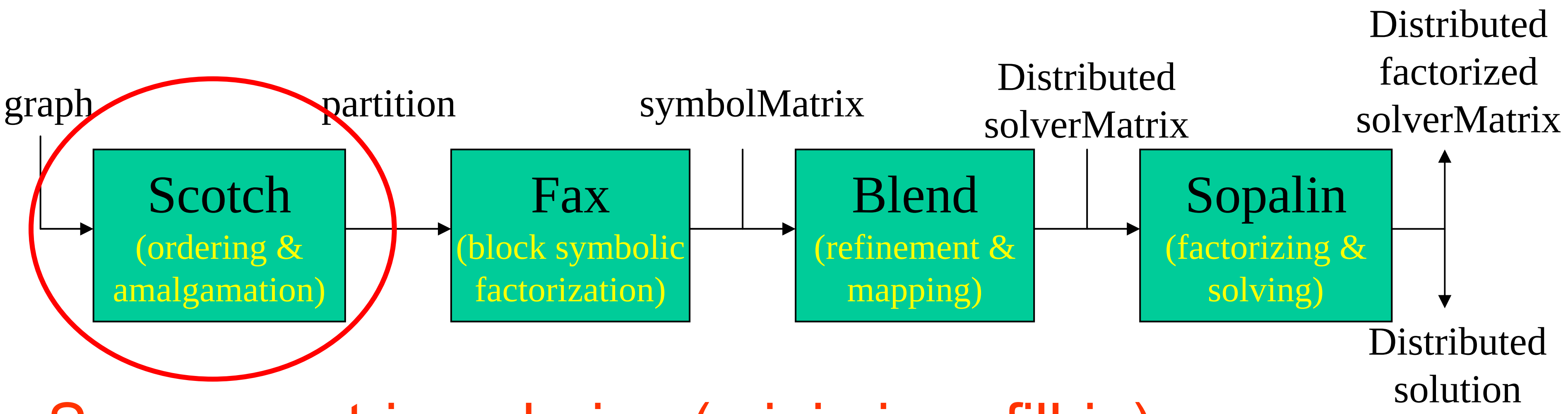

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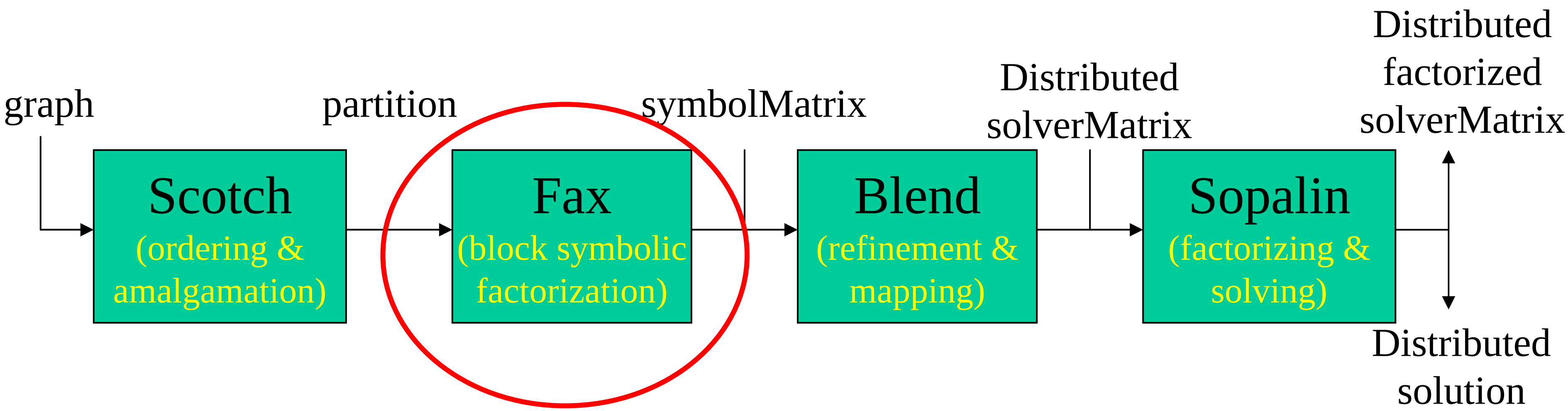

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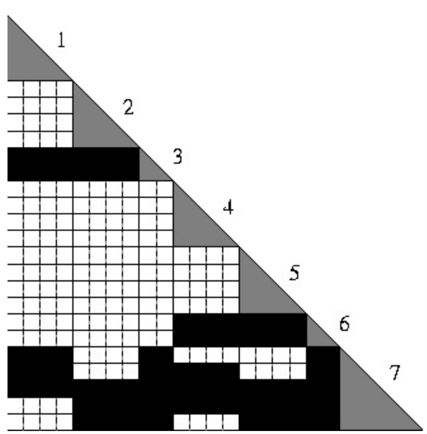

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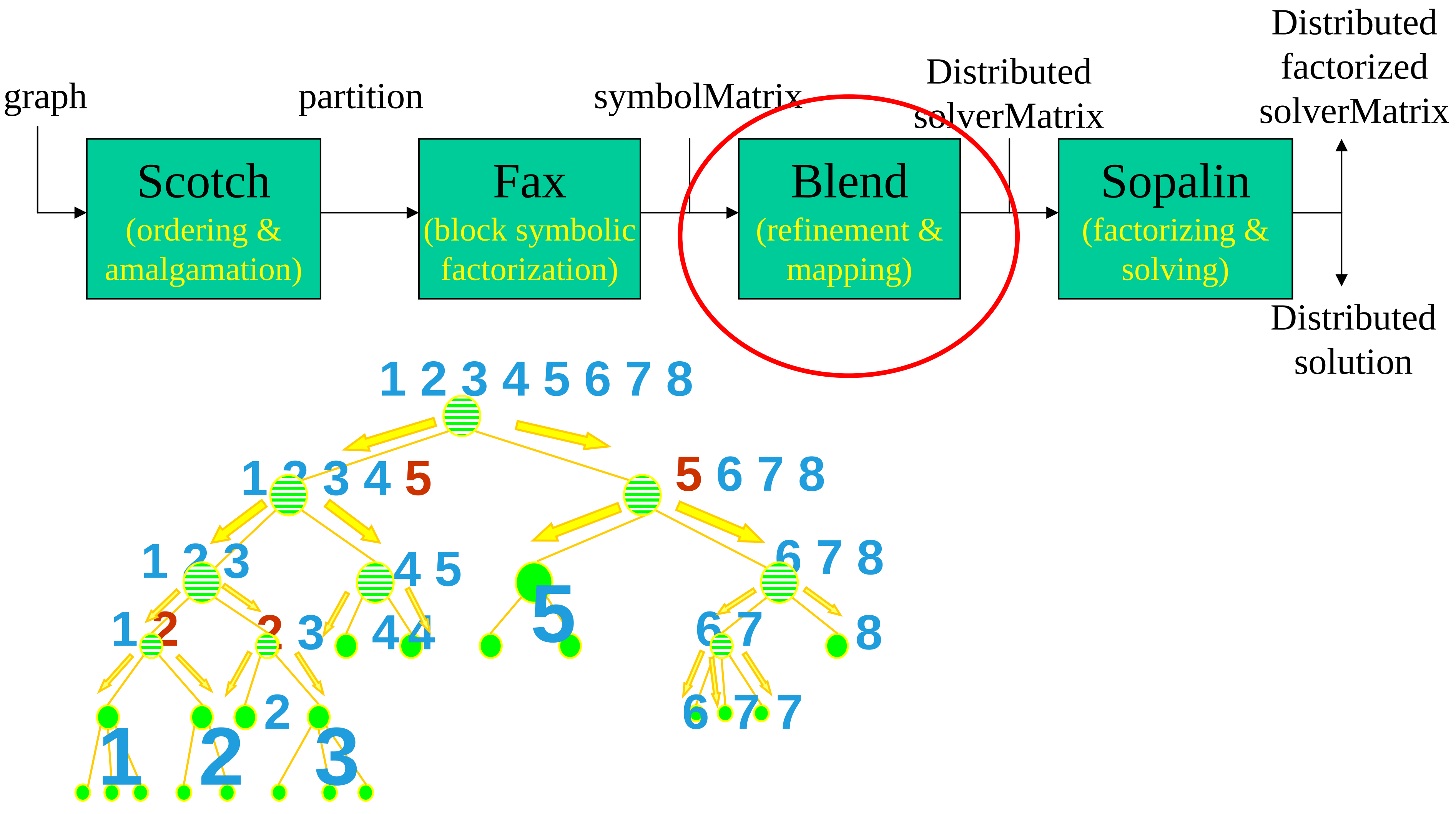

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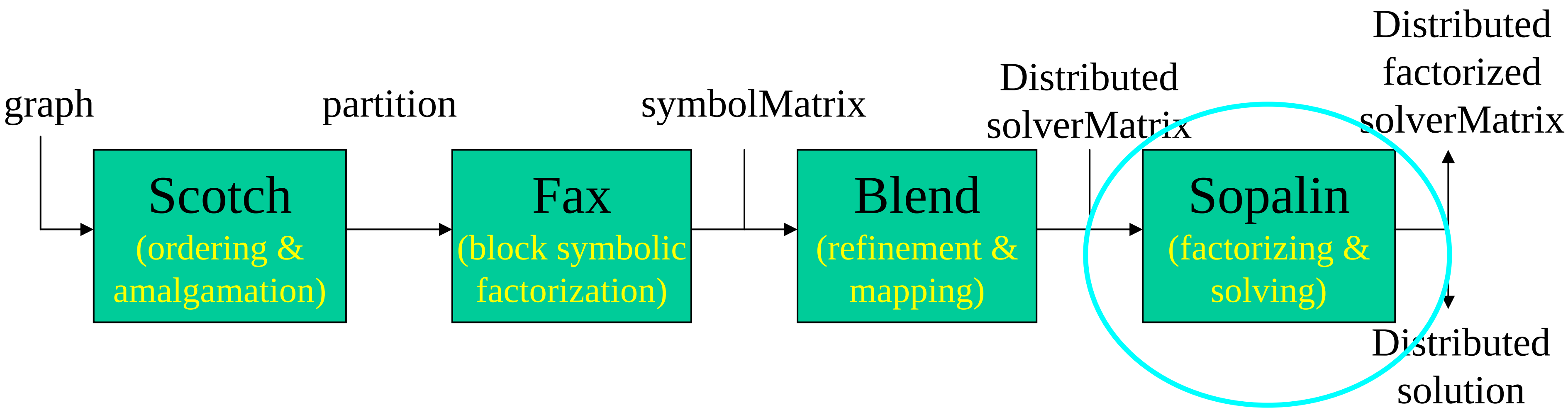

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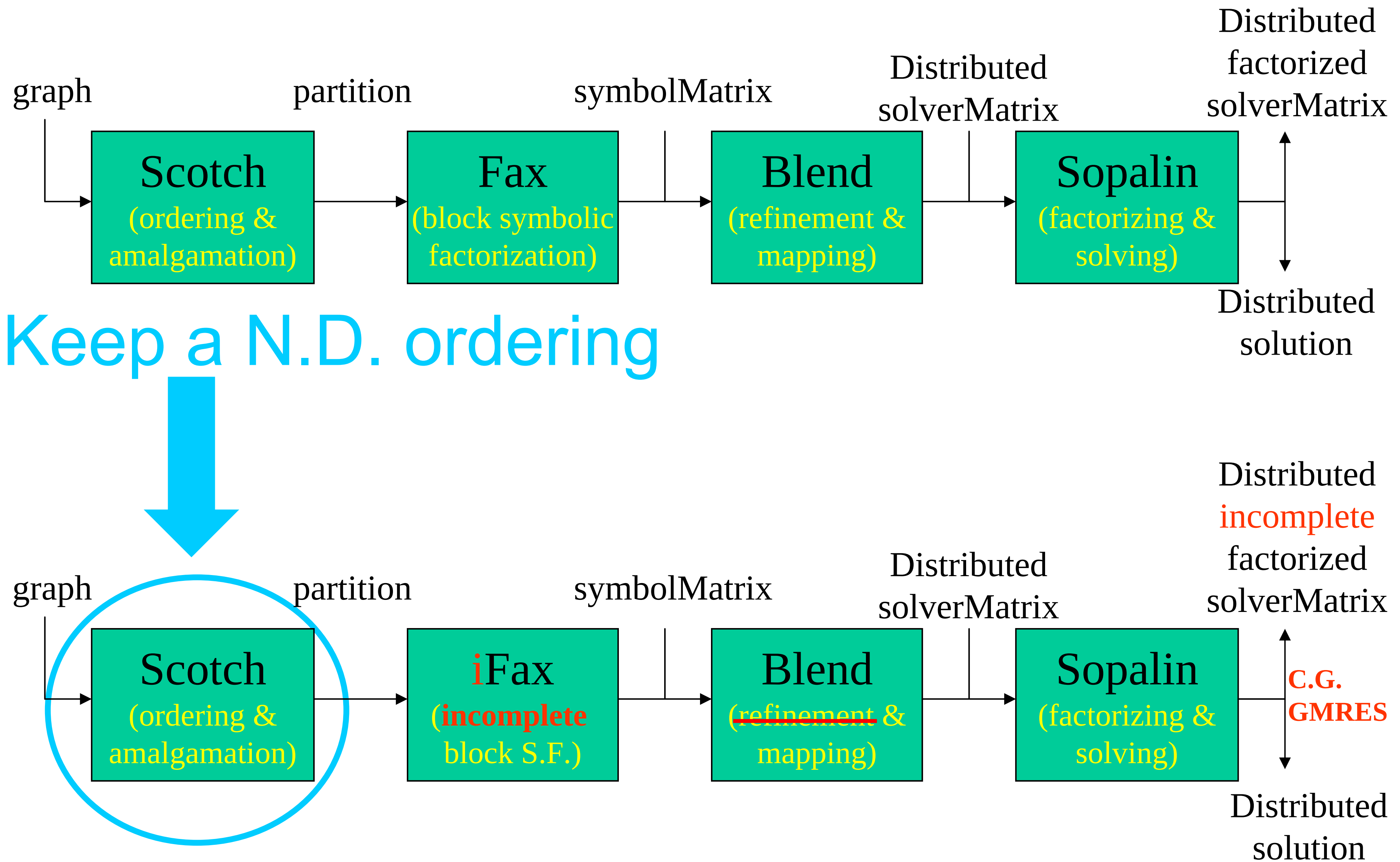
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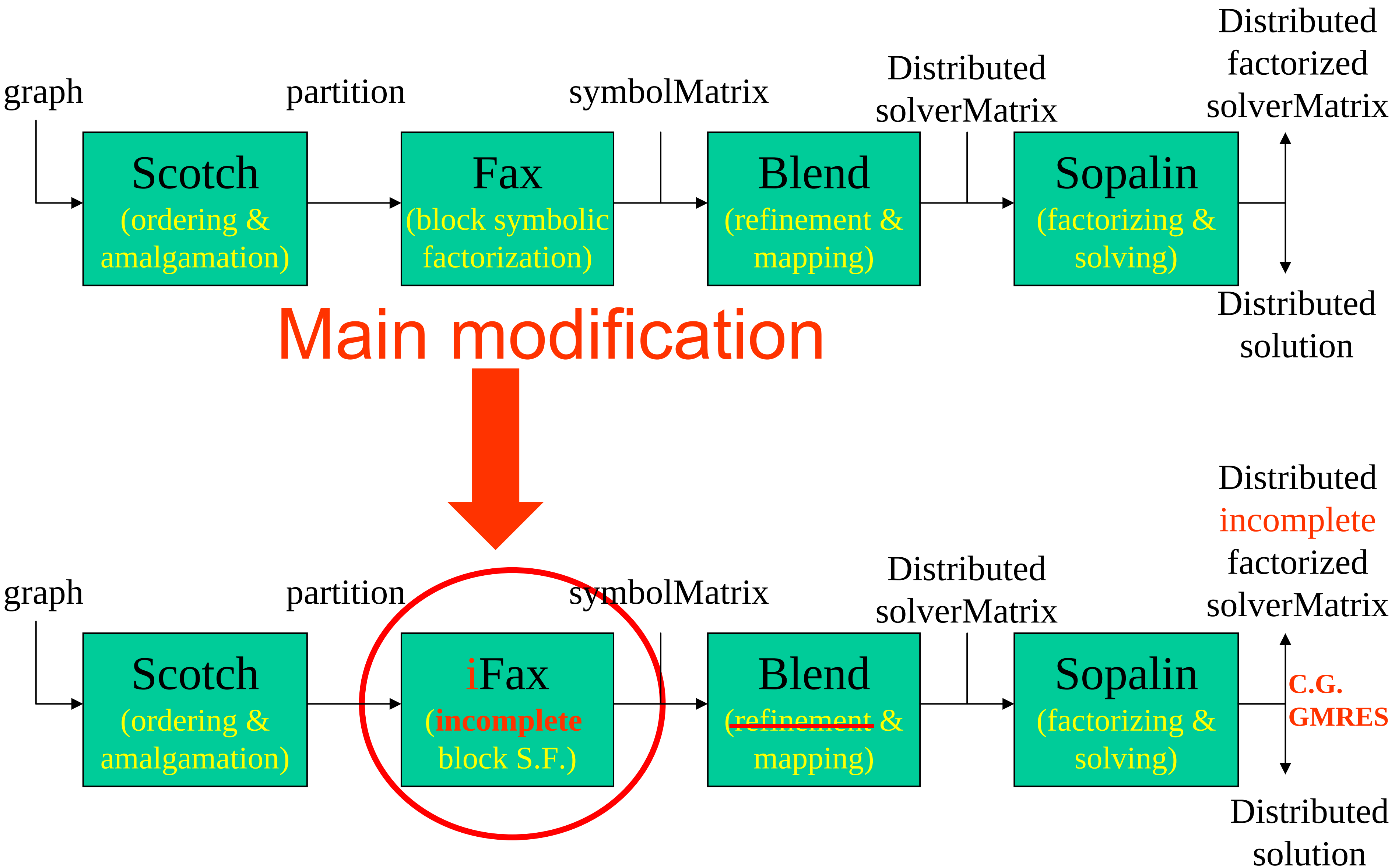
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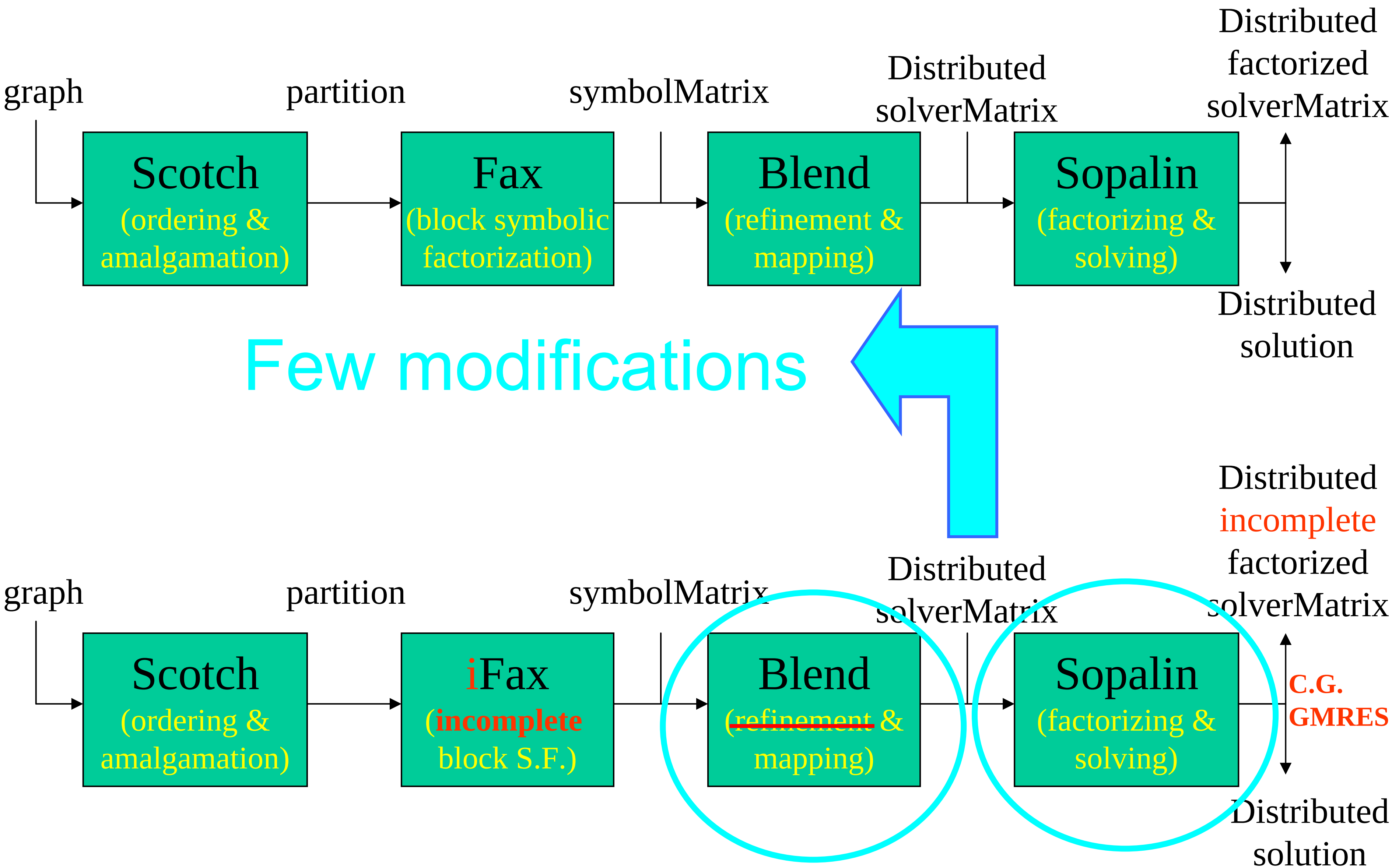
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Direct Factorization techniques

- Ordering to minimize fill-in and allow // is based upon ND
- Partition of supernodes P is found in 0(nnzA).
- ***Q(G,P)→Q(G,P)*=Q(G*,P)***
  => linear in number of blocks!
- Dense block structures
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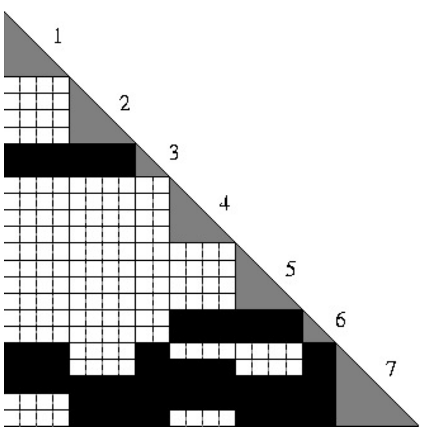

Direct Factorization techniques

⇒ Manage parallelism induced by sparsity (block elimination tree).

⇒ Split and distribute the dense blocks in order to take into account the potential parallelism induced by dense computations .