4^{th} **International Workshop on Parallel Matrix Algorithms and Applications**

Convergence of iterative solvers for the method coupling Finite elements and integral representation

F. Ben Belgacem, UTC, France.

N. Gmati, LAMSIN/ENIT, Tunisia

F. Jelassi, LAMSIN/ENIT, Tunisia.

B. Philippe, IRISA/INRIA-Rennes, France

Rennes - September 7,8,9 2006

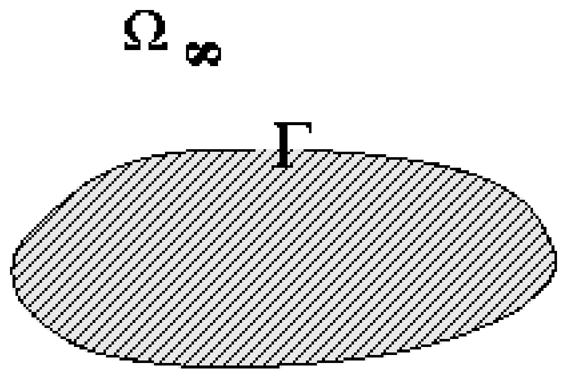
Outline

- Introduction
- The method coupling Finite elements and integral representation
- Iterative solvers - Convergence results
- Numerical results

Problems in unbounded domains

- Acoustic scattering
  (**Helmholtz equations**)
- Electromagnetic scattering
  (**Maxwell equations**)
- Eddy current models
- Plate bending
  (**Harmonic Bilaplacian**)
- Fluid mechanics
  (**Stationary Stokes problem**)

Acoustic scattering problems



$$
\begin{cases}
\Delta u + k^2 u = 0 & \text{in} \quad \Omega_\infty \\
\dfrac{\partial u}{\partial \nu} = f & \text{on} \quad \Gamma \\
\text{Sommerfeld Radiation Condition}
\end{cases}
\tag{1}
$$

Transparent boundary conditions

- **Approximated boundary conditions** **Local B.C.**

- **Radiation condition at finite distance**
  (Bayliss-Turkell, Bendali-Halpern, ...)

- **Infinite elements**
  (J.Emson, ...)

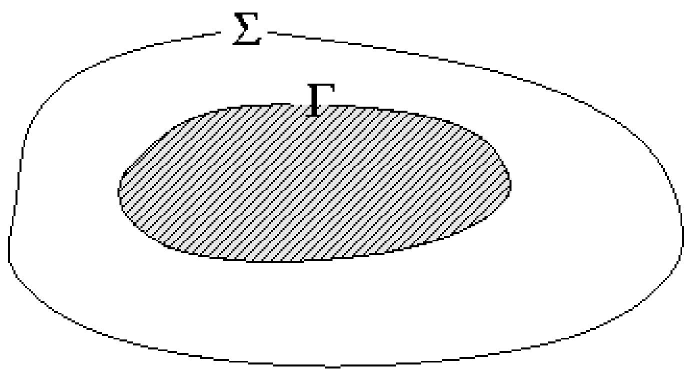
- **M.E.I method**
  (K.K.MEI, F.Collino)

- **Higher order boundary conditions**
  (Enguist-Majda, F.Collino, D.Givoli)

Transparent boundary conditions

- **Exact boundary conditions: Non Local B.C.**
- **Integral equations**
  (Johnson-Nedelec, Joly-Kern, Hamdi, Bendali, ...)
- **The method coupling finite elements and integral equations**
  (Nédélec, Bendali, ...)
- **Perfectly Matching Layer**
  (Beranger, Bécache-Bonnet Bendhia, ...)
- **Dirichlet to Neumann boundary condition**
  (Morgan-Mei, Lenoir-Tounsi, ...)
- **The method coupling finite elements and integral representation**
  (Jami-Lenoir, Liu-Jin, ...)

The method coupling finite elements and integral representation (CEFRI)

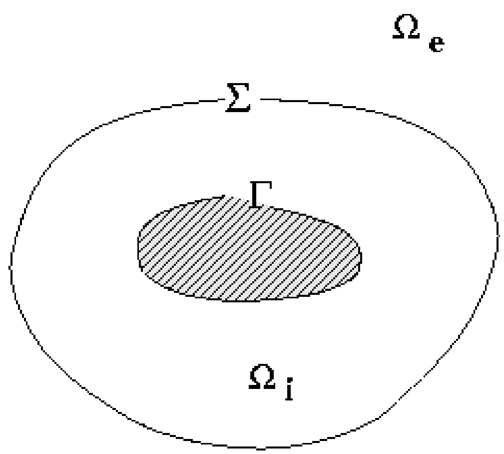


$$\begin{cases} \Delta u + k^2 u & = & 0 & \text{in} & \Omega_i \\ \dfrac{\partial u}{\partial \nu} & = & f & \text{on} & \Gamma \\ \left(\dfrac{\partial}{\partial \nu} + \lambda\right) u & = & T_\lambda u & \text{on} & \Sigma \end{cases} \tag{2}$$

$$(T_\lambda u)(x) = \int_\Gamma u(y) \frac{\partial K}{\partial \nu}(x-y) - \frac{\partial u}{\partial \nu} K(x-y) \mathrm{d}\gamma \tag{3}$$

$$K(x-y) = \left(\frac{\partial}{\partial \nu_x} + \lambda\right) G(x-y), \qquad Im(\lambda) \neq 0 \tag{4}$$

CEFRI = Domain decomposition method with overlapping



$$\begin{cases} \Delta u + k^2 u = 0 & (\Omega_i) \\ \dfrac{\partial u}{\partial \nu} = f & (\Gamma) \\ \left(\dfrac{\partial}{\partial \nu} + \lambda\right) u = \left(\dfrac{\partial}{\partial \nu} + \lambda\right) v & (\Sigma) \end{cases} \qquad \begin{cases} \Delta v + k^2 v = 0 & (\Omega_i \cup \Omega_e) \\ [\dfrac{\partial v}{\partial \nu}] = \dfrac{\partial u}{\partial \nu} & (\Gamma) \\ [v] = u & (\Gamma) \\ \text{Sommerfeld Radiation condition} & \end{cases}$$

$$v = \int_\Gamma u(y) \frac{\partial G}{\partial \nu}(x-y) - \frac{\partial u}{\partial \nu} G(x-y) \mathrm{d}\gamma \quad (\Sigma)$$

$$\left(\frac{\partial}{\partial \nu} + \lambda\right) v = T_\lambda u(x) = \int_\Gamma u(y) \frac{\partial K}{\partial \nu}(x-y) - \frac{\partial u}{\partial \nu} K(x-y) \mathrm{d}\gamma \quad (\Omega_i \cup \Omega_e)$$

The method coupling finite elements and integral representation (CEFRI)

- **The variational formulation:**

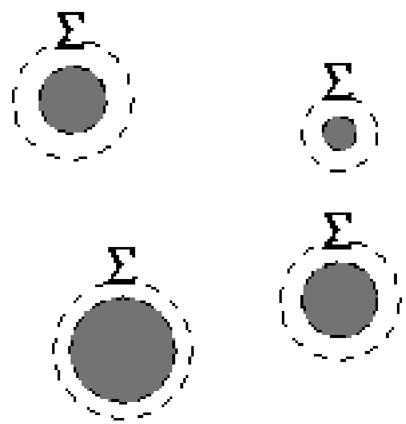
$$\begin{cases} \text{Find } u \in H^1(\Omega_i), u \neq 0 \text{ such that} \\ a(u,v) - c(u,v) = l_\Gamma(v) - l_\Sigma(v), \quad \forall v \in H^1(\Omega_i), \end{cases}$$

$$\begin{aligned}
a(u,v) &= \int_{\Omega_i} \left(\nabla u(x) \nabla \overline{v}(x) - k^2 u(x) \overline{v}(x) \right) dx - ik \int_\Sigma u(x) \overline{v}(x) d\sigma(x) \\
c(u,v) &= \int_\Sigma \overline{v}(x) \Big(\int_\Gamma u(y) \partial_n K(x-y) d\gamma(x) \Big) d\sigma(x) \\
l_\Gamma(v) &= \int_\Gamma f(x) \overline{v}(x) d\gamma(x), \\
l_\Sigma(v) &= \int_\Sigma \overline{v}(x) \Big(\int_\Gamma f(y) K(x-y) d\gamma(x) \Big) d\sigma(x)
\end{aligned}$$

- **Finite element discretization - The linear system:**

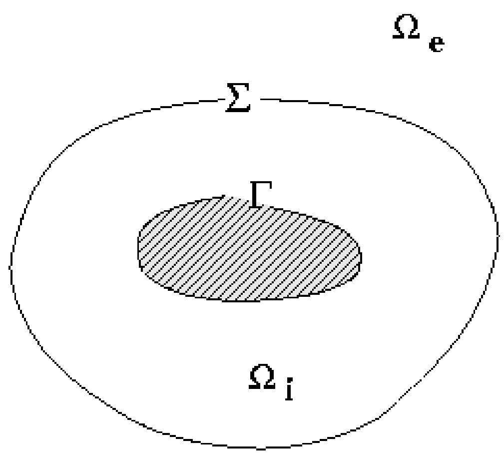
$$(A - C)U = F$$

Remarks about the CEFRI method



- A reduced Computational domain.
- No singularities in the Green kernal.
- A Full block : Σ and Γ are coupled
- The matrix (A-C) is complex, non hermitian and ill-conditioned

The Schwarz method :Overlapping decomposition method



$$\begin{cases} \Delta u^{n+1} + k^2 u^{n+1} = 0 & (\Omega_i) \\ \dfrac{\partial u^{n+1}}{\partial \nu} = f & (\Gamma) \\ \dfrac{\partial u^{n+1}}{\partial \nu} + \lambda u^{n+1} = \dfrac{\partial v^n}{\partial \nu} + \lambda v^n & (\Sigma) \end{cases}$$

$$\begin{cases} \Delta v^{n+1} + k^2 v^{n+1} = 0 & (\Omega_i \cup \Omega_e) \\ [\dfrac{\partial v^{n+1}}{\partial \nu}] = \dfrac{\partial u^{n+1}}{\partial \nu} & (\Gamma) \\ [v^{n+1}] = u^{n+1} & (\Gamma) \\ \text{Sommerfield Radiation Condition} \end{cases}$$

$$v^{n+1} = \int_\Gamma u^{n+1}(y) \frac{\partial G}{\partial \nu}(x-y) - \frac{\partial u^{n+1}}{\partial \nu} G(x-y) \mathrm{d}\gamma \quad (\Sigma)$$

$$\left(\frac{\partial}{\partial \nu} + \lambda \right) v^{n+1} = T_\lambda u^{n+1}(x)$$

The Schwarz method

$$\begin{cases} \Delta u^{n+1} + k^2 u^{n+1} = 0 & \text{in} \quad \Omega_i \\ \dfrac{\partial u^{n+1}}{\partial \nu} = f & \text{on} \quad \Gamma \\ \dfrac{\partial u^{n+1}}{\partial \nu} + \lambda u^{n+1} = T_\lambda u^n & \text{on} \quad \Sigma \end{cases} \tag{5}$$

- **Finite element discretization - Linear system:**

$$(A - C)U = F \quad \rightarrow \quad AU^{n+1} = CU^n + F$$

- **Convergence result:**

- **Convergence only if** $\rho(A^{-1}C) < 1$.

- **This condition is satisfied if** Σ **is sufficiently far from** Γ.

CEFRI preconditioned by Schwarz method

- **Initial linear system:**

$$(A - C)U = F$$

- **After preconditionning by A**

$$(I_{N_\Omega} - B_\Omega)U = A^{-1}F.$$
$$B_\Omega = A^{-1}C = B_1 B_2$$

Where $B_1 = A^{-1}P_\Sigma^t M_\Sigma$ (N_Ω rows, and N_Σ columns)

and $B_2 = G_n M_\Gamma P_\Gamma$ (N_Σ rows and N_Ω columns).

The Steklov-Poincaré interface equation

- **We introduce an auxilary unknown defined on Σ:**

$$\varphi = T_\lambda u, \qquad \text{on} \qquad \Sigma. \tag{6}$$

$$(\partial_n - ik)u = \varphi \qquad \text{on} \qquad \Sigma. \tag{7}$$

- **Variational formulation of the problem on u:**

$$\begin{cases} \text{Find } u \in H^1(\Omega_c), u \neq 0 \text{ such that} \\ a(u,v) - \displaystyle\int_\Gamma \varphi(x)\overline{v}(x)d\gamma(x) = l_\Gamma(v), \quad \forall v \in H^1(\Omega_c), \end{cases}$$

- **Finite element discretization - Linear system:**

$$AU - P_\Sigma^t M_\Sigma \Phi = F_\Gamma$$

$$\Phi = G_n M_\Gamma P_\Gamma U - H.$$

The Steklov-Poincaré interface equation

- **The reduced equation on Σ:**

$$(I_{N_\Sigma} - B_\Sigma)\Lambda = \Lambda_{inc} \tag{8}$$

$$\text{where} \qquad B_\Sigma = B_2 B_1 \tag{9}$$

- **Convergence result:**

- **Remark: Except eventually for the eigenvalue 0, $B_\Omega = B_1 B_2$ and $B_\Sigma = B_2 B_1$ have the same eignenvalues, with the same multiplicity. They have eventually few eigenvalues out of the unit disk.**

- **The GMRES method applied to $(I_{N_\Sigma} - B_\Sigma)$ and to $(I_{N_\Omega} - B_\Omega)$ converge.**

Estimation of superlinear convergence

[Kerkhoven-Saad '02] : superlinear convergence for linear operators with eigenvalues clustered in neighborhoods of 1. When expressed for matrices :

Theorem : If $A = I - B \in \mathbb{R}^{n \times n}$ is non singular and diagonalizable, then for GMRES :

$$\|r_k\| \leq c_k(\rho) \left(\frac{n\rho}{(1-\rho)k} \right)^k ,$$

for any $0 < \rho < 1$ and where $\lim_{\infty} c_k = c$.

Theorem : If $A = I - B \in \mathbb{R}^{n\times n}$ is non singular, with p eigenvalues of B out of the unit disk, then for GMRES and for $k \geq p$:

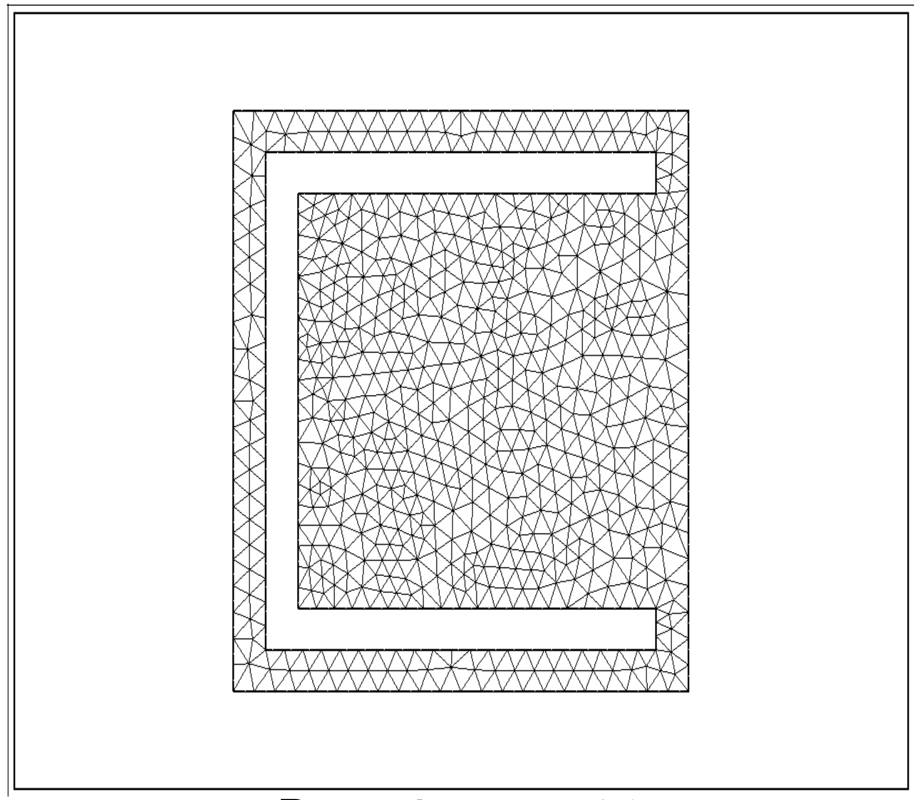
$$\|r_k\| \leq K \|r_{k-p}^{reduced}\|$$

where $r_k^{reduced}$ is the residual corresponding to an application of GMRES on the system projected onto the invariant subspace excluding the exterior eigenvalues.

We have $K \leq \gamma \; cond(X)$ where

- $X = [X_1, X_2]$ with X_1 basis of the invariant subspace corresponding to the p largest eigenvalues of B and X_2 basis of the supplementary invariant subspace,
- $\gamma = \max_{\lambda_j \notin D} \prod_{\lambda_i \in D} \frac{|\lambda_j - \lambda_i|}{|1-\lambda_i|}$, where (λ_i) are the eigenvalues of B.

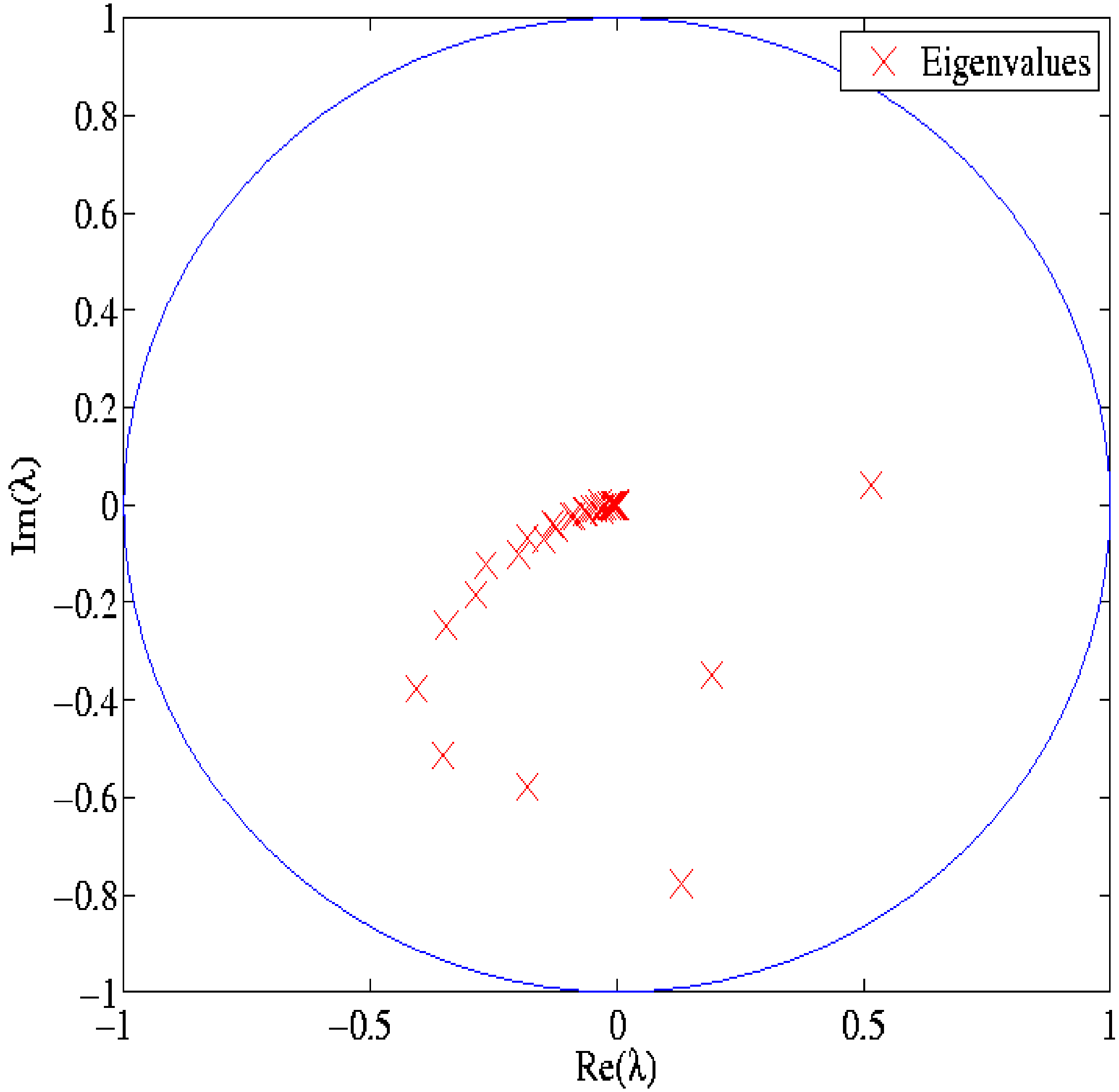
Numerical results



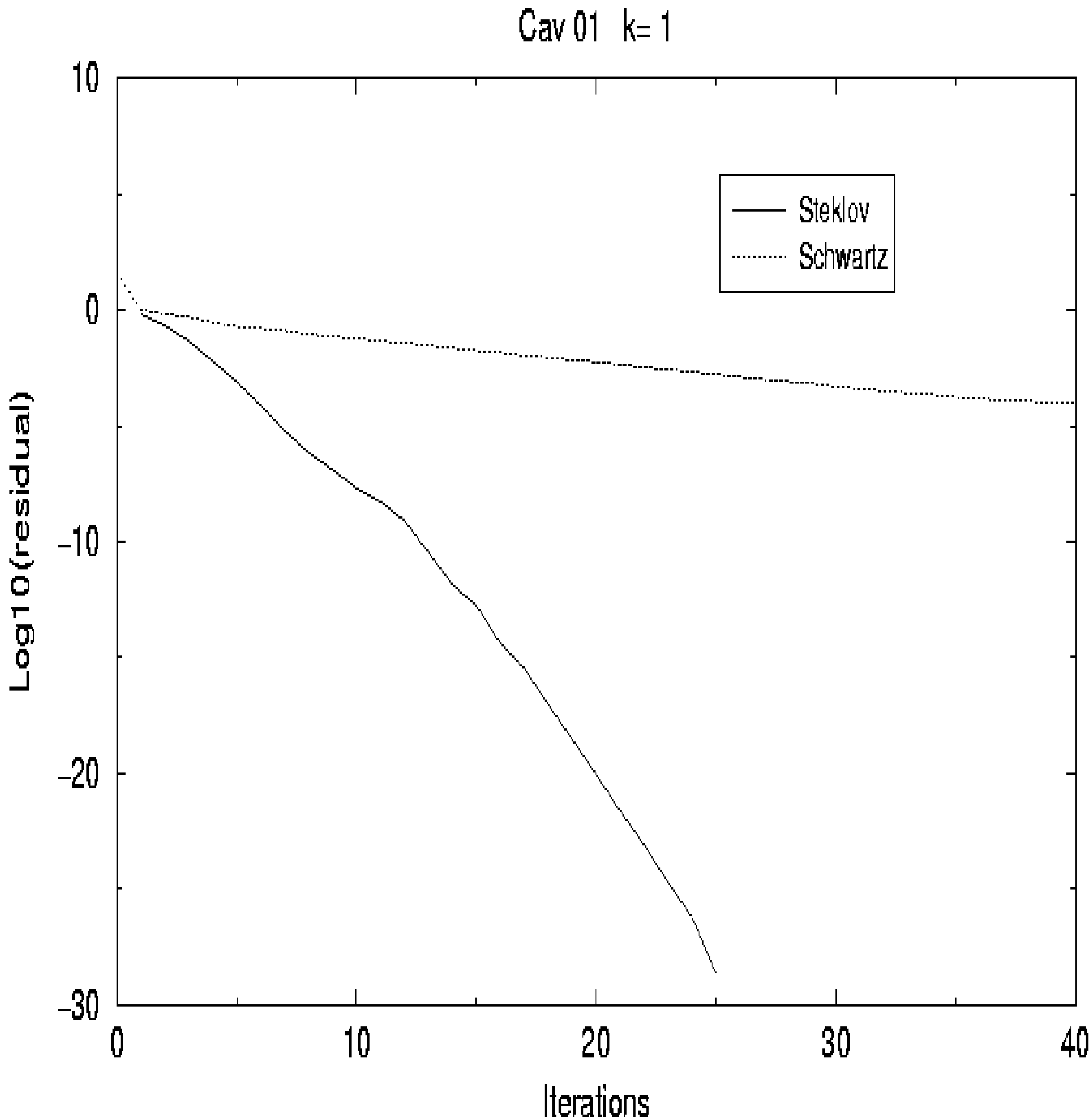
Domain : cav01

d	$NBEL$	N_Ω	N_Σ	h
0.1	1251	734	88	0.06

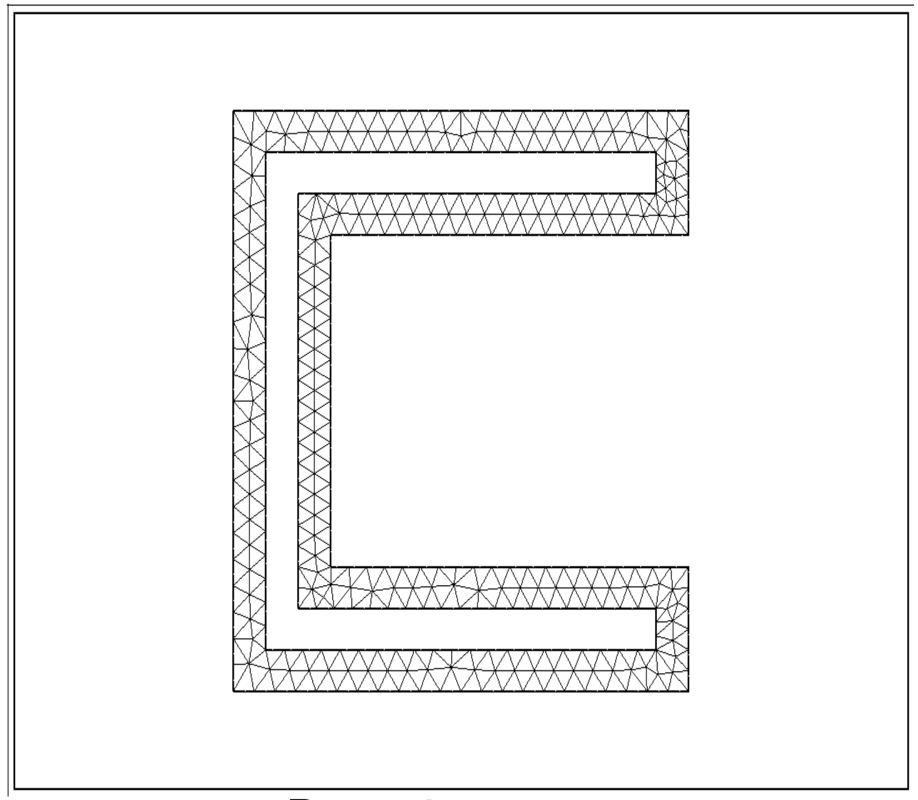
The numerical results are obtained thanks to MELINA, a finite elements code (D.Martin, Rennes).



k=1, Eigenvalues of $B_\Omega = B_\Sigma = A^{-1}C$

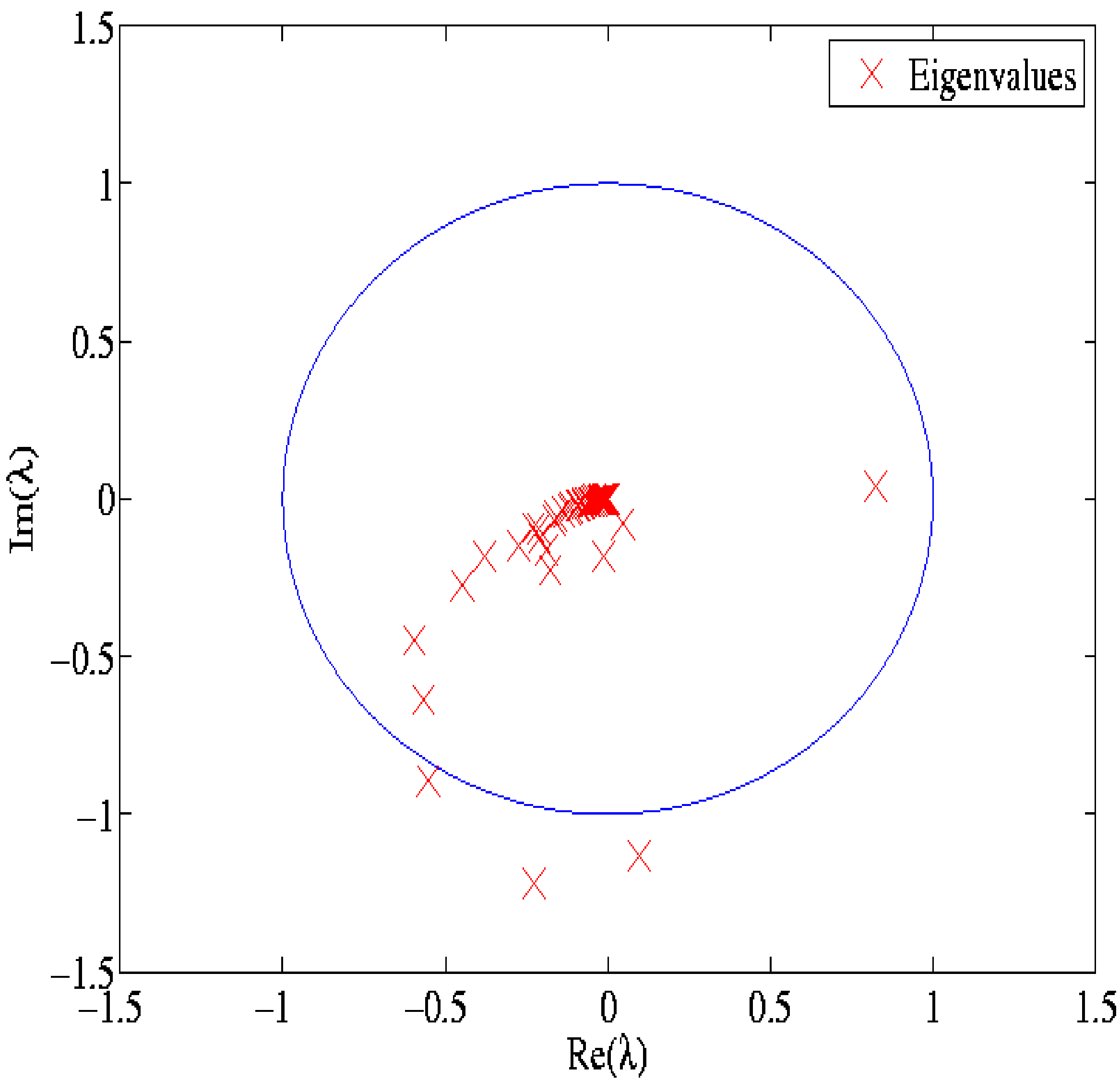


Convergence of GMRES (Steklov equation) and Schwarz method

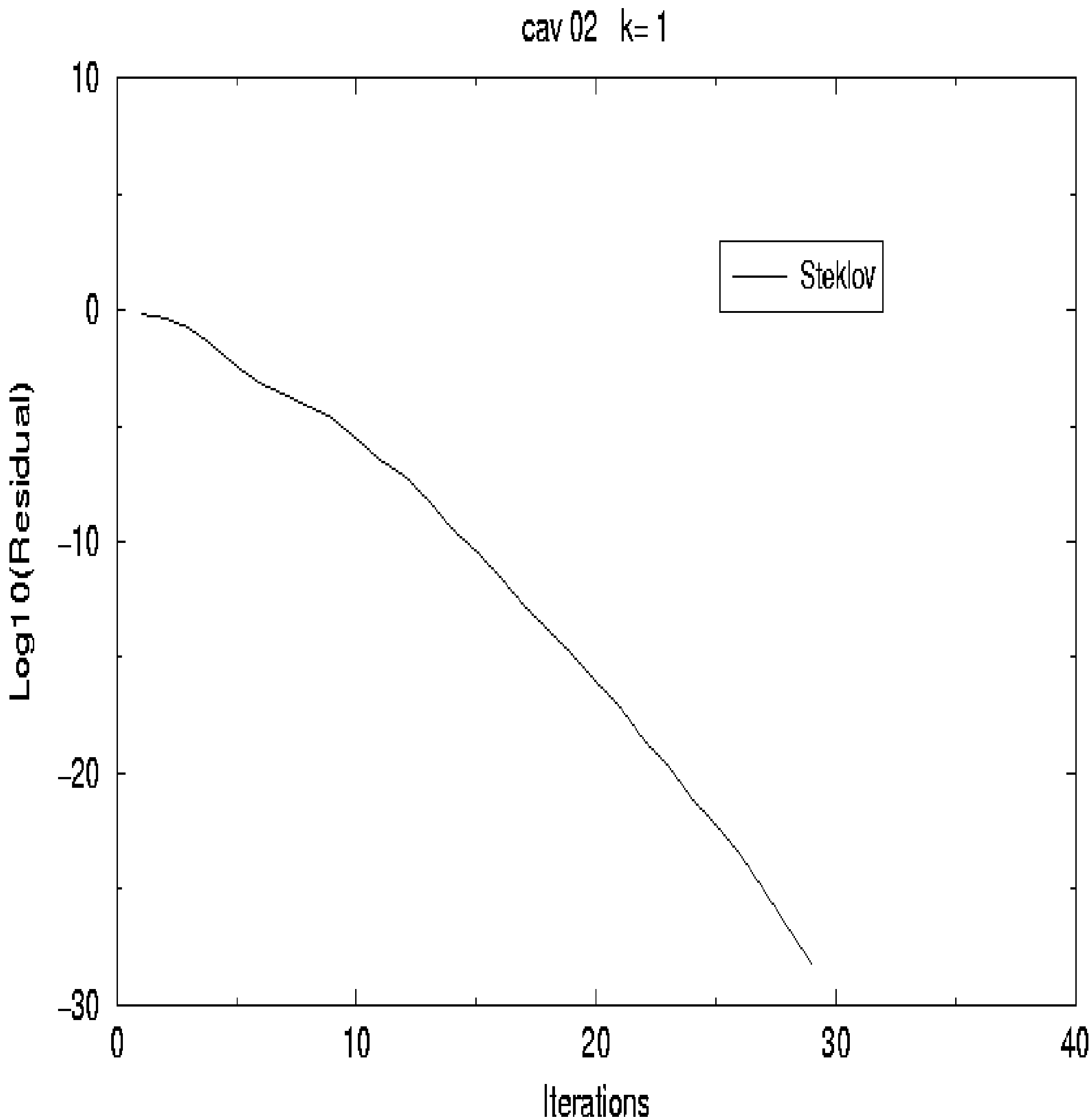


Domain: cav02

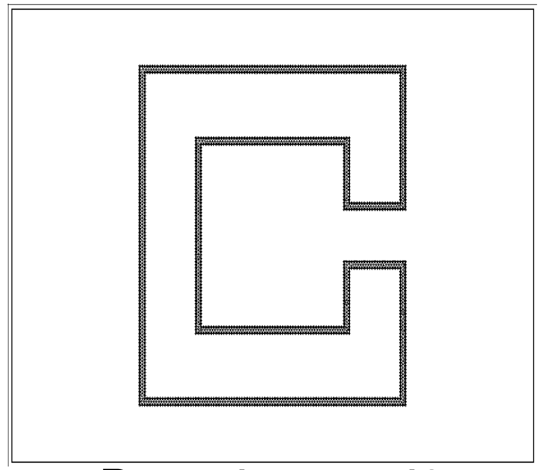
d	$NBEL$	N_Ω	N_Σ	h
0.1	543	403	134	0.06



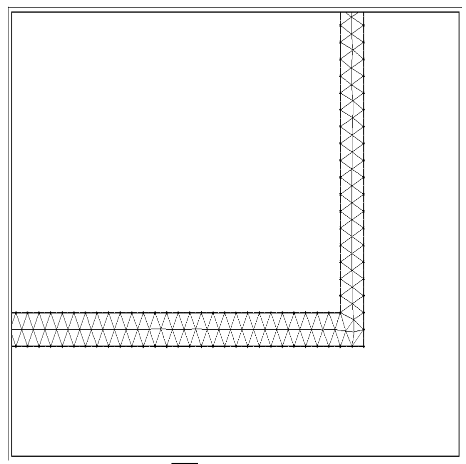
k=1, Eigenvalues of $B_\Omega = B_\Sigma = A^{-1}C$



Convergence of GMRES

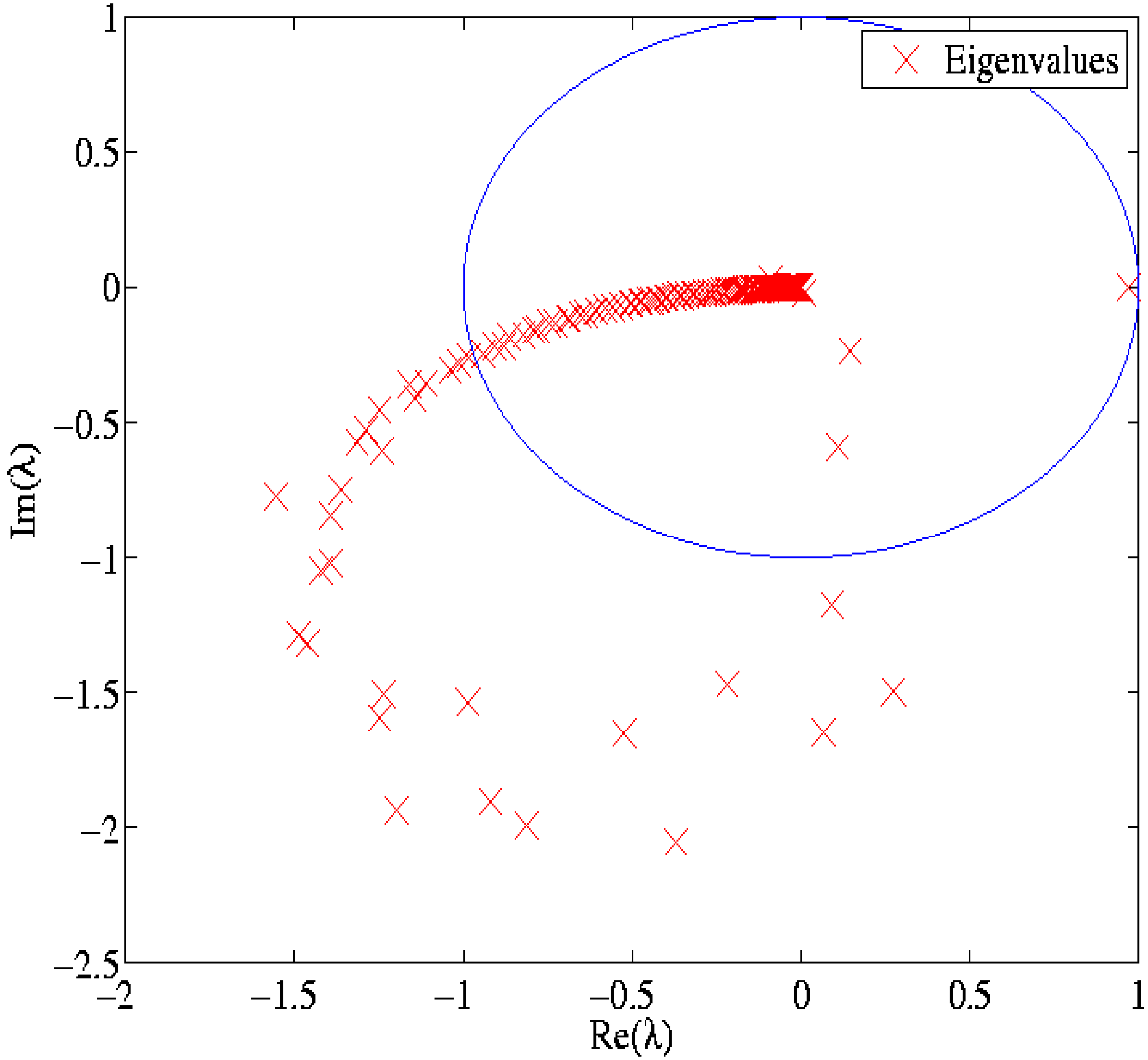


Domain : mai2

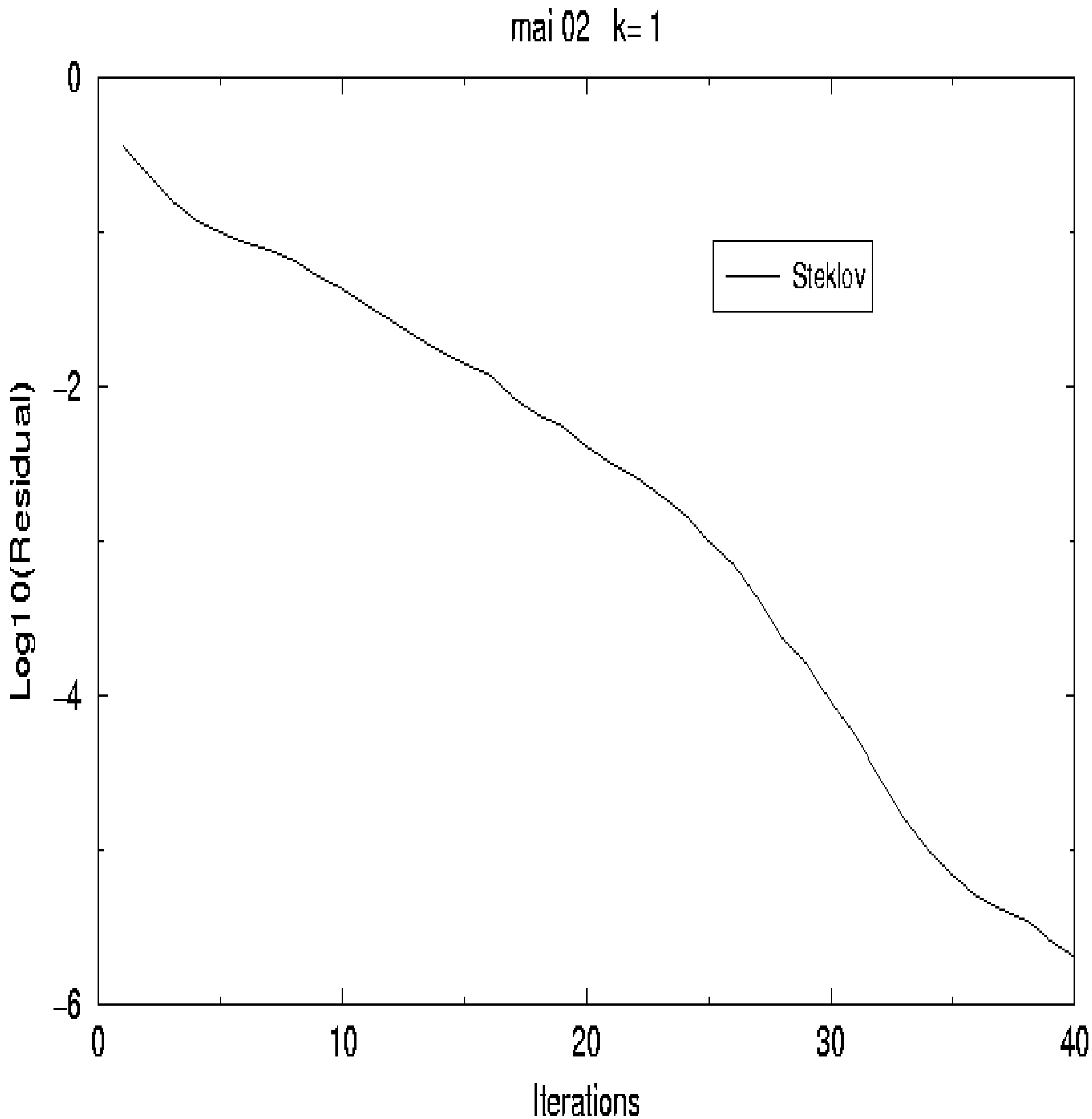


Zoom

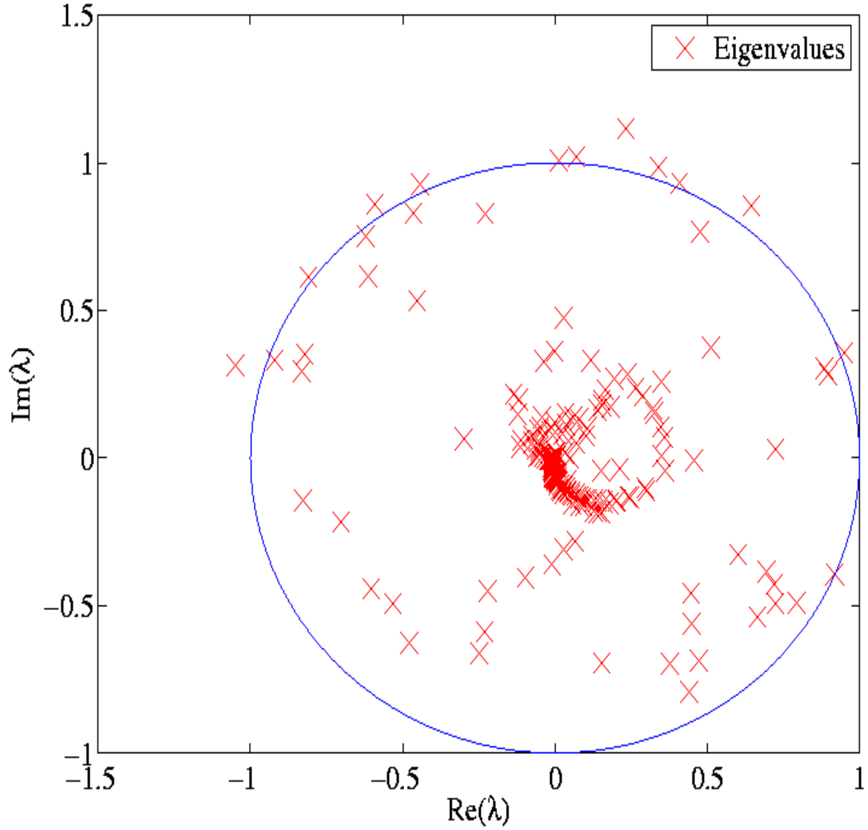
d	$NBEL$	N_{Ω}	N_{Σ}	h
0.01	2630	1963	656	0.005



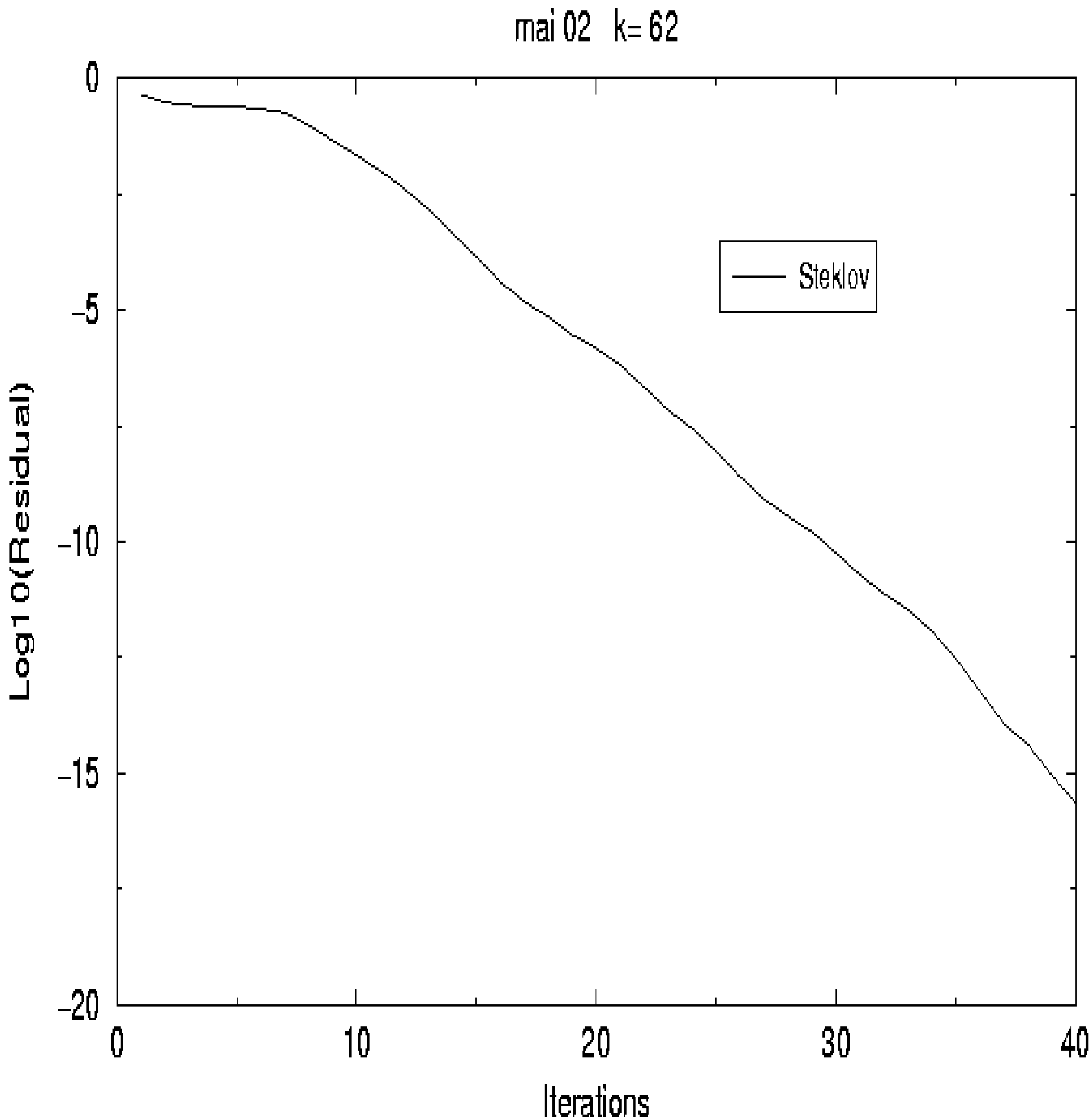
k=1, Eigenvalues of $B_\Omega = B_\Sigma = A^{-1}C$



Convergence of GMRES



k=62, Eigenvalues of $B_{\Omega} = B_{\Sigma} = A^{-1}C$



Convergence of GMRES