A new cement to glue nonconforming grids with Robin interface conditions: the finite element case

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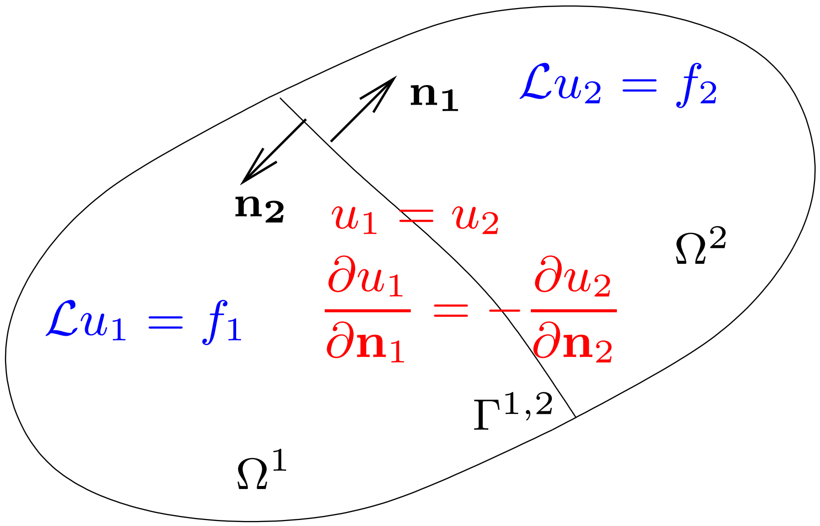
Error estimates : with Yvon Maday and Frédéric Nataf (Université Paris 6)

Projection algorithm : with Martin Gander (Université de Genève)

Model elliptic problem

$$\mathcal{L}u = u - \Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$



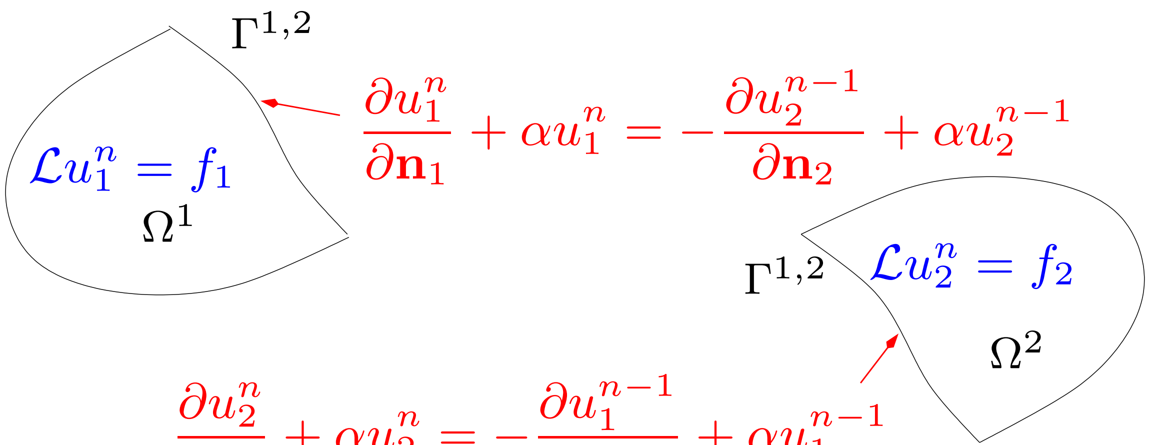
Robin transmission conditions:

Let $\alpha > 0$ such that the physical interface conditions are equivalent to

$$\frac{\partial u_1}{\partial \mathbf{n}_1} + \alpha u_1 = -\frac{\partial u_2}{\partial \mathbf{n}_2} + \alpha u_2$$

$$\frac{\partial u_2}{\partial \mathbf{n}_2} + \alpha u_2 = -\frac{\partial u_1}{\partial \mathbf{n}_1} + \alpha u_1$$

Schwarz type algorithm



$$\frac{\partial u_1^n}{\partial \mathbf{n}_1} + \alpha u_1^n = -\frac{\partial u_2^{n-1}}{\partial \mathbf{n}_2} + \alpha u_2^{n-1}$$

$$\frac{\partial u_2^n}{\partial \mathbf{n}_2} + \alpha u_2^n = -\frac{\partial u_1^{n-1}}{\partial \mathbf{n}_1} + \alpha u_1^{n-1}$$

choose α in order to optimize the convergence rate

Problem : **How to discretize these conditions with finite elements and nonmatching grids?**

Advantages

1. parallel generation of meshes.
2. locally structured meshes, fast and independent solvers in the subdomains.
3. local adaptive meshes.

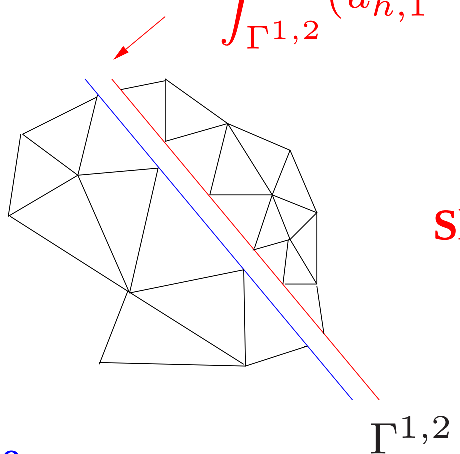
Finite elements: **Mortar Method** (Dirichlet/Neumann)

(Bernardi/Maday/Patera (1989), Ben Belgacem (1993), Braess/Dahmen (1998))

$$\int_{\Gamma^{1,2}} (u_{h,1} - u_{h,2})\psi_2 = 0, \ \forall \psi_2 \text{ Mortar function}$$

Ω_1
Master

Neumann Problem
with C.L. obtained from
the residual on the slave side



Ω_2
Slave

The mortar element method with Robin transmission conditions

On each subdomain Ω^k, we have a triangular mesh $\mathcal{T}_k$ whose triangles have maximal diameters h_k. **The meshes are constructed in an independent manner.**

Associated with the mesh $\mathcal{T}_k$, we consider the spaces of piecewise linear Lagrange finite elements

$$\begin{aligned} Y_h^k &= \{v_{h,k} \in \mathcal{C}^0(\overline{\Omega}^k),\ \ v_{h,k|T} \in \mathbb{P}_1(T),\ \forall T \in \mathcal{T}_k\}, \\ X_h^k &= \{v_{h,k} \in Y_h^k,\ v_{h,k|\partial\Omega^k \cap \partial\Omega} = 0\}. \end{aligned}$$

The space of traces over each $\Gamma^{k,\ell}$ of elements of Y_h^k is denoted by $\mathcal{Y}_h^{k,\ell}$.

The discretization parameter : $h = \max\limits_{1 \leq k \leq K} h_k$.

The discrete problem

Find $(\underline{u}_h, \underline{p}_h) \in (\prod_k X_h^k) \times \left(\prod_k \prod_{\ell / \Gamma^{k,\ell} \neq \emptyset} W^{k,\ell}\right)$ satisfying the continuity condition across the interface $\Gamma^{k,\ell}$:

$$\int_{\Gamma^{k,\ell}} ((p_{h,k} + \alpha u_{h,k}) - (-p_{h,\ell} + \alpha u_{h,\ell}))\psi_{k,\ell} = 0, \ \forall \psi_{k,\ell} \in W^{k,\ell}$$

such that $\forall \underline{v}_h = (v_1, ... v_K) \in \prod_k X_h^k,$

$$\sum_{k=1}^{K} \int_{\Omega_k} (\nabla u_{h,k} \nabla v_k + u_{h,k} v_k)\, dx - \sum_{k=1}^{K} \int_{\partial \Omega_k} p_{h,k} v_k ds = \sum_{k=1}^{K} \int_{\Omega_k} f_k v_k dx$$

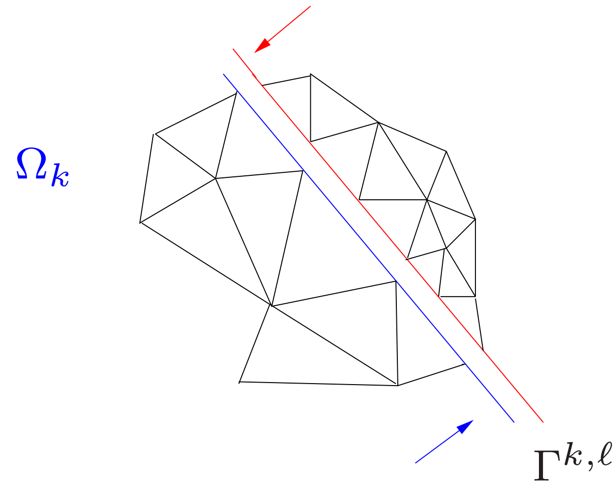
where $W^{k,\ell}$ **is a suitably defined test function space**

$\Rightarrow$ **stability, optimal error estimates**

A symmetric way: no master, no slave

(for a finite volume scheme : Achdou/Japhet/Maday/Nataf (2002))

$$\int_{\Gamma^{k,\ell}} (p_{h,\ell} + \alpha u_{h,\ell})\psi_{\ell,k} = \int_{\Gamma^{k,\ell}} (-p_{h,k} + \alpha u_{h,k})\psi_{\ell,k},\ \forall \psi_{\ell,k} \in W^{\ell,k}$$

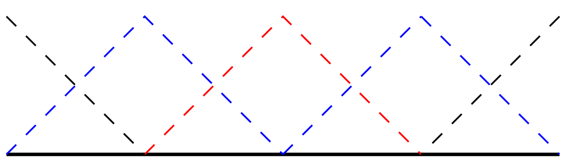


$$\int_{\Gamma^{k,\ell}} (p_{h,k} + \alpha u_{h,k})\psi_{k,\ell} = \int_{\Gamma^{k,\ell}} (-p_{h,\ell} + \alpha u_{h,\ell})\psi_{k,\ell},\ \forall \psi_{k,\ell} \in W^{k,\ell}$$

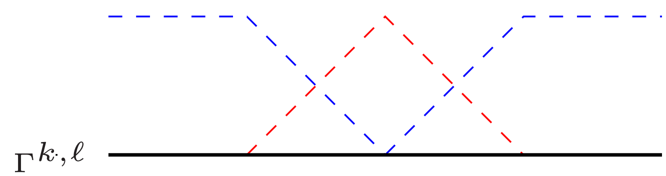
The test functions for the weak continuity condition

As for more usual mortar methods, the space $W^{k,\ell}$ is constructed as a subspace of $\mathcal{Y}_h^{k,\ell}$.

As for the nonoverlapping mortar method, **one must remove degrees of freedom at the endpoints of $\Gamma^{k,\ell}$**.



Test function of $\mathcal{Y}_h^{k,\ell}$



Test function of $W^{k,\ell}$

Theorem : Stability and well-posedness.

Theorem : Best approximation error.

Assume that the solution u of the original problem is in $H^2(\Omega) \cap H_0^1(\Omega)$.

- If $u_{|\Omega_k} \in H_*^2(\Omega_k),$ and $\alpha = \frac{c}{h},$ then

$$\|\underline{u}_h - \underline{u}\|_* \leq ch \sum_{k=1}^{K} \|u\|_{H^2(\Omega_k)}$$

- if $u_{|\Omega_k} \in H_*^2(\Omega_k),$ $p_k = \frac{\partial u}{\partial \mathbf{n}_k} \in H^{\frac{3}{2}}(\Gamma^{k,\ell})$ and $\alpha = \dfrac{c}{h^\gamma},$ $0 \leq \gamma \leq 1,$

$$\|\underline{u}_h - \underline{u}\|_* = O(h^{1+\gamma}|log(h)| + h).$$

The discrete algorithm

Let (u_k^n, p_k^n) be a discrete approximation of $(\underline{u}, \underline{p})$ in Ω_k at step n. Then, (u_k^{n+1}, p_k^{n+1}) is the solution in $X_h^k \times \prod_{\ell/\Gamma^{k,\ell}\neq\emptyset} W^{k,\ell}$ of

$$\int_{\Omega_k} \left(\nabla u_k^{n+1} \nabla v_k + u_k^{n+1} v_k\right) dx - \int_{\partial\Omega_k} p_k^{n+1} v_k ds = \int_{\Omega_k} f_k v_k dx, \quad \forall v_k \in X_h^k$$

$$\int_{\Gamma^{k,\ell}} (p_k^{n+1} + \alpha u_k^{n+1}) \psi_{k,\ell} = \int_{\Gamma^{k,\ell}} (-p_\ell^n + \alpha u_\ell^n) \psi_{k,\ell}, \quad \forall \psi_{k,\ell} \in W^{k,\ell}$$

Theorem

Let $\alpha h \leq c$, for some constant c small enough. Then, the discrete problem has a unique solution $(\underline{u}_h, \underline{p}_h) \in \mathcal{V}_h$. The discrete algorithm is well posed and converges in the sense that for $1 \leq k \leq K$,

$$\lim_{n \longrightarrow \infty} \left(\|u_k^n - u_{h,k}\|_{H^1(\Omega_k)} + \sum_{\ell \neq k} \|p_{k,\ell}^n - p_{h,k,\ell}\|_{H_*^{-\frac{1}{2}}(\Gamma^{k,\ell})} \right) = 0.$$

Extensions

- in 2D for finite elements approximation of degree $M \leq 13$
- in 3D for P1 finite element

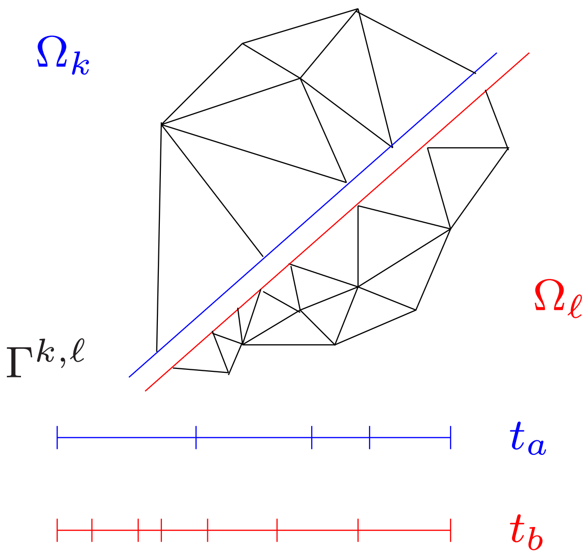
proofs in *A new cement to glue nonconforming grids with Robin interface conditions: the finite element case*, C. Japhet, Y. Maday, F. Nataf, to be submitted.

Implementation: an efficient way to perform the projections in 2D

Based on the method in Gander/Halpern/Nataf (2003)

Let $\{\psi_i^k\}$ (resp. $\{\psi_j^\ell\}$) the basis functions of $W^{k,\ell}$ (resp. $W^{\ell,k}$)

How to compute $M_{i,j} = \int_{\Gamma^{k,\ell}} \psi_i^k \psi_j^\ell$?




```
for i=1:n-1
    tm=tb(i);
    while ta(j+1)<tb(i+1)
        M(i:i+1,j:j+1)=M(i:i+1,j:j+1)+Mortarformula;
        j=j+1;
        tm=ta(j);
    end;
    M(i:i+1,j:j+1)=M(i:i+1,j:j+1)+Mortarformula;
end;
```

$\Rightarrow$ **Linear complexity algorithm without an additional grid**

Schwarz accelerated by Krylov

The Optimized Schwarz algorithm is a Jacobi algorithm applied to the interface problem

$$D\lambda = b$$

λ_i : discretization of $(\frac{\partial}{\partial \mathbf{n}_i} + \alpha)(u_i)$ on the interface $\Gamma_{i,j}$

$(D\lambda)_i$: discretization of the jump $(\frac{\partial}{\partial \mathbf{n}_i} + \alpha)(u_i - u_j)$ on $\Gamma_{i,j}$

$\Rightarrow$ **Replace Jacobi algorithm by a Krylov type algorithm to accelerate convergence**

Numerical Results

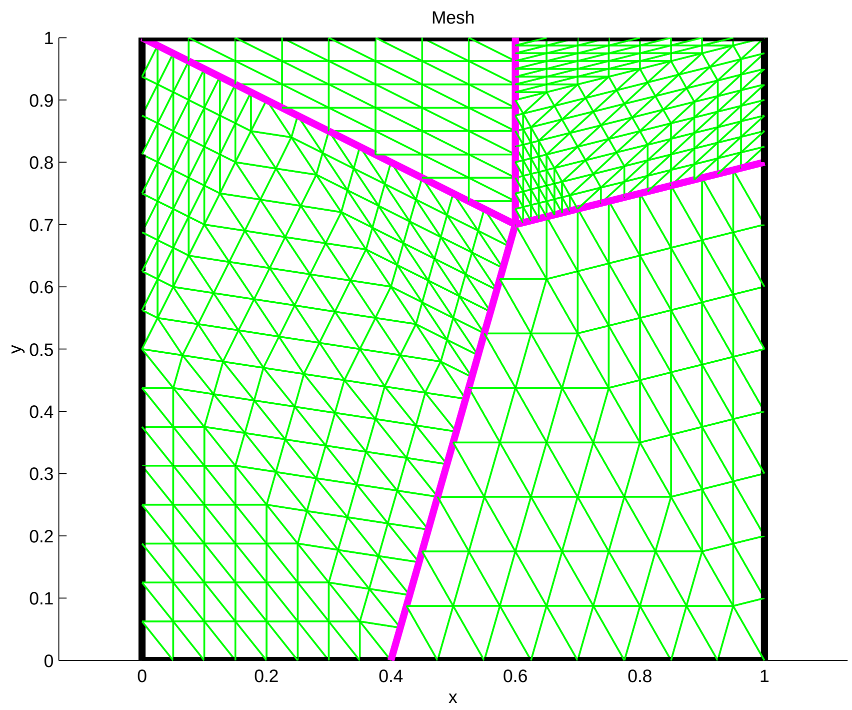
Exact solution $u(x, y) = x^3y^2 + sin(xy)$

Stopping criterion : the transmission conditions jumps must be smaller than 10^{-8}

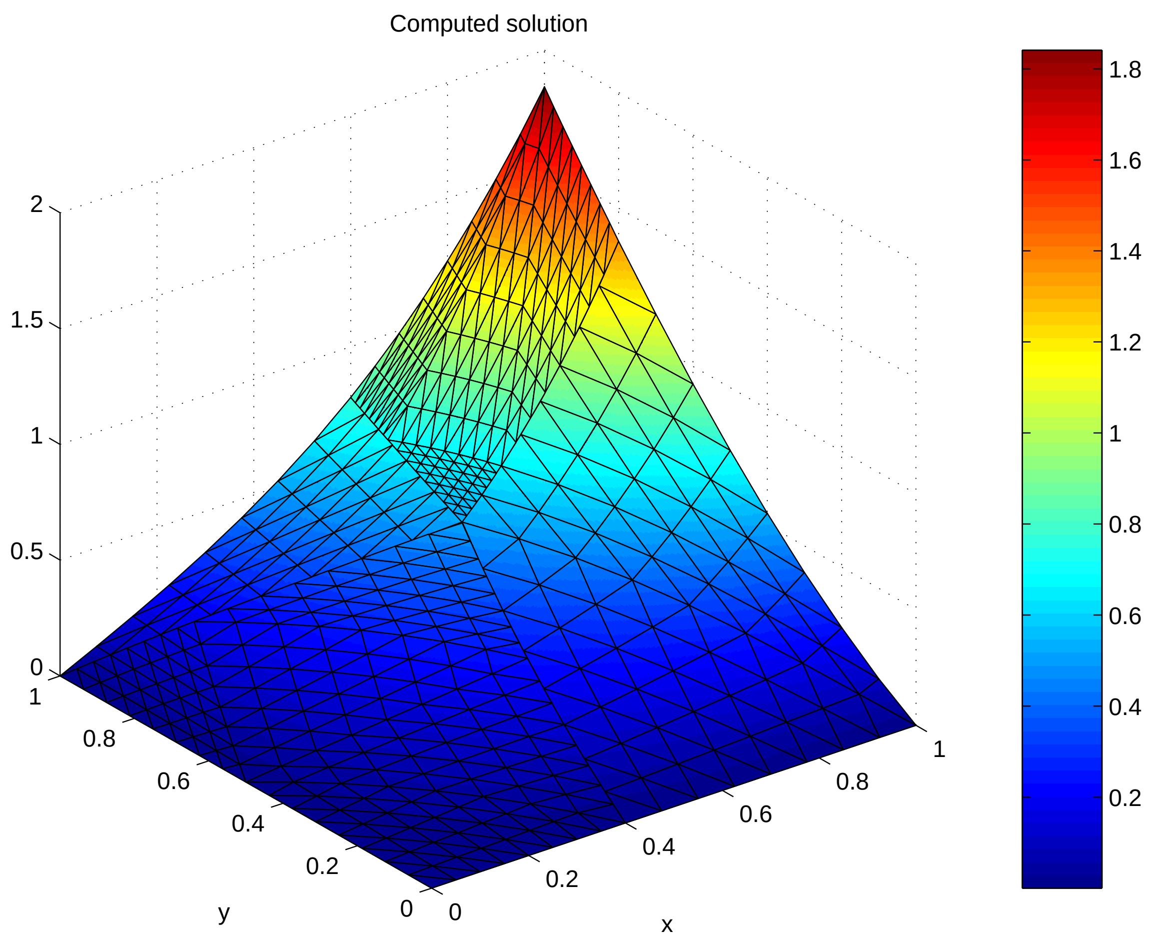
Robin Parameter α : constant or minimize the convergence rate

An example of computed solution

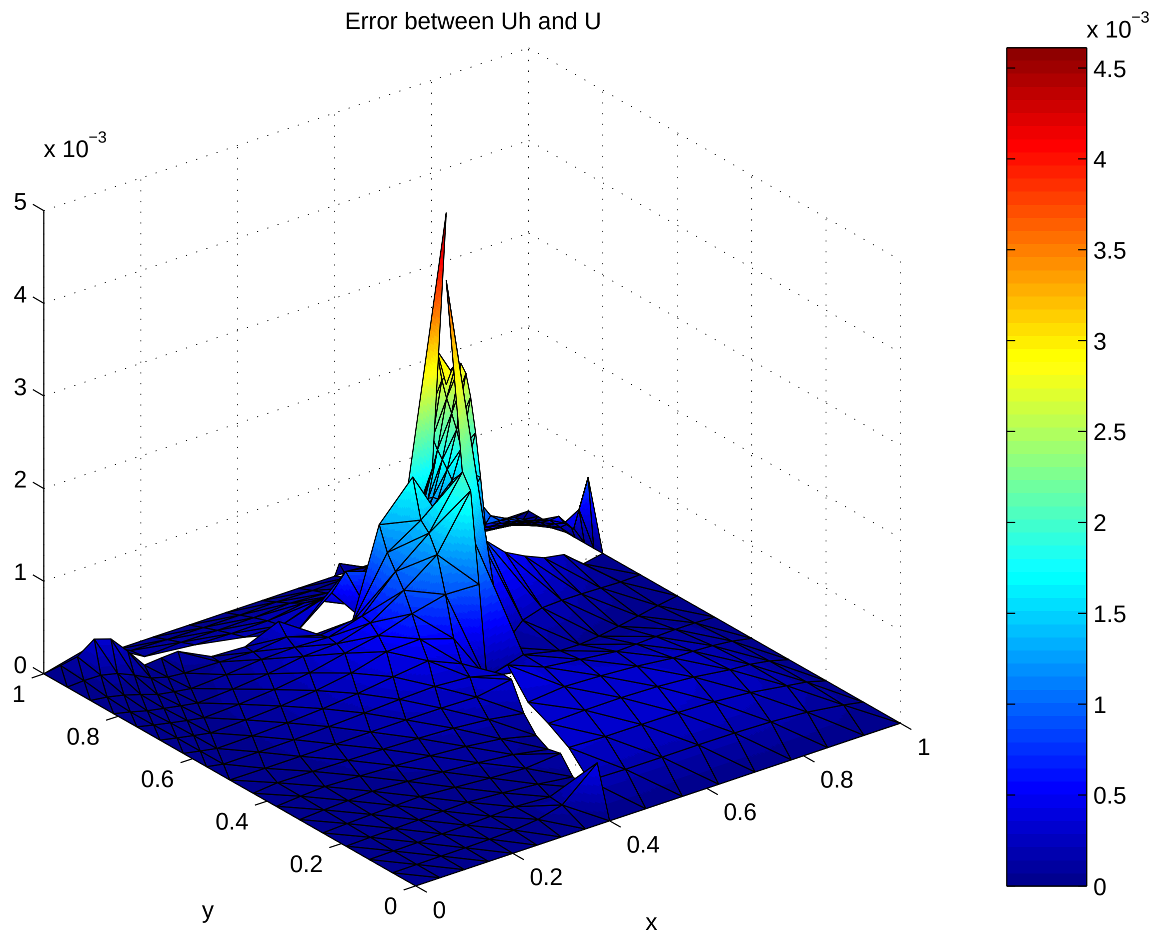
Nonconforming 2×2 domain decomposition, $\alpha = 10$



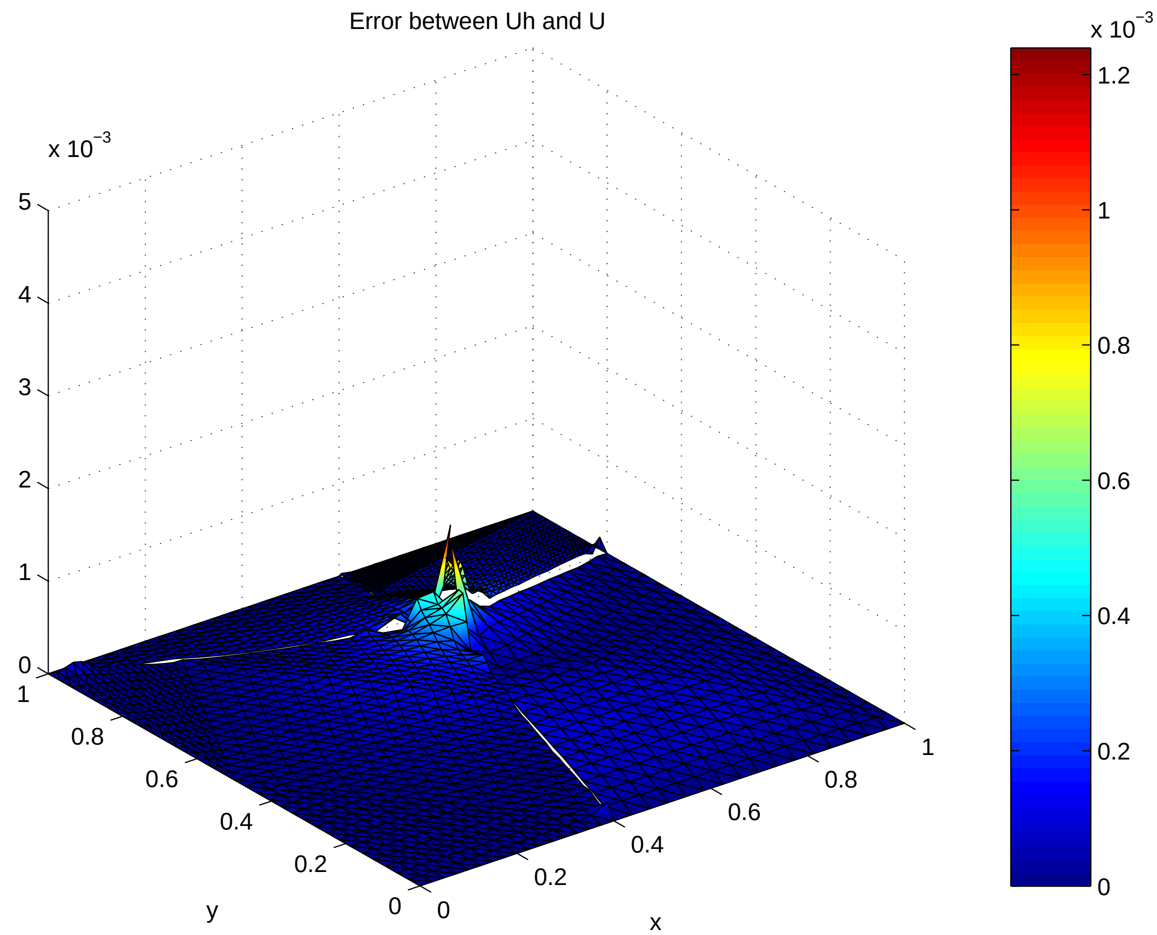
computed solution



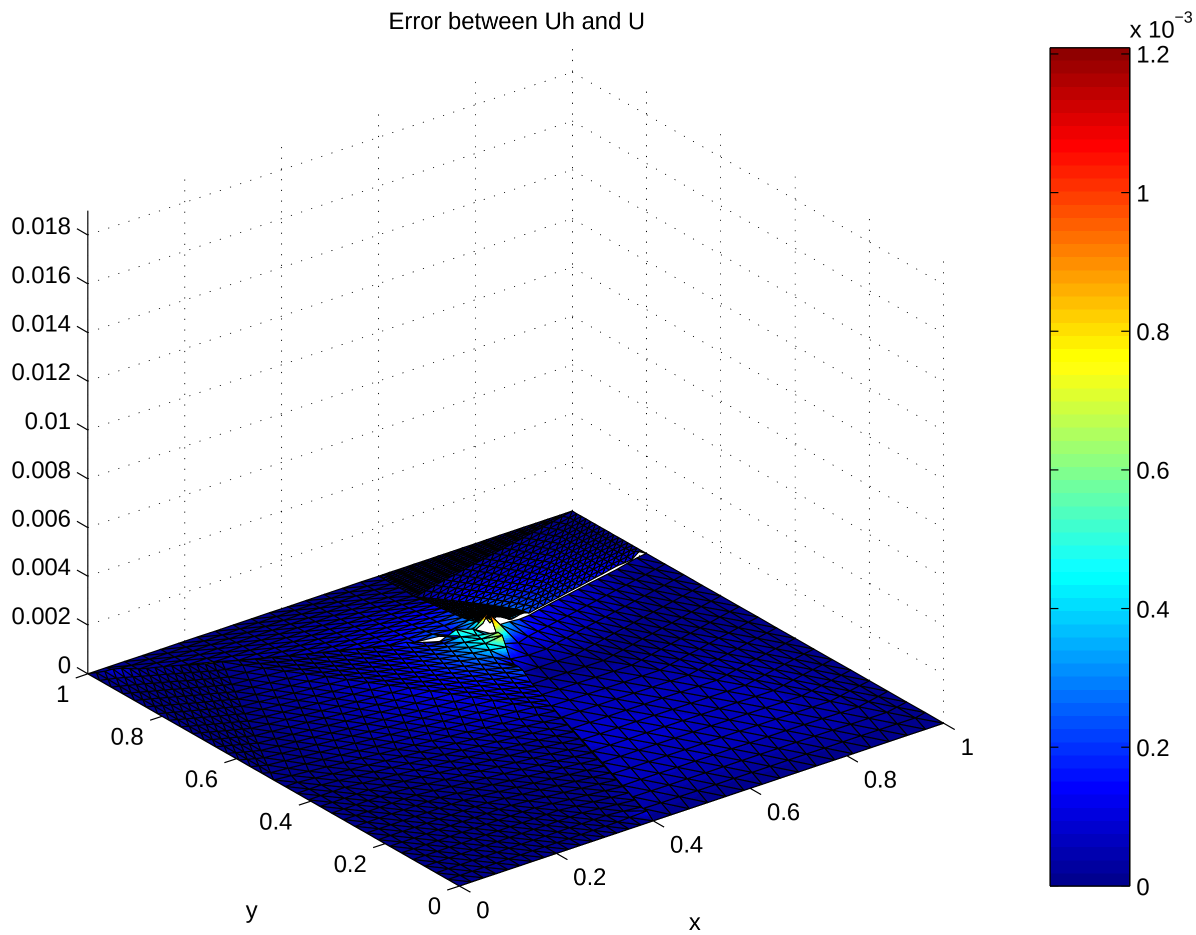
Error $|u - u_h|$ mesh size $h = 0.1$



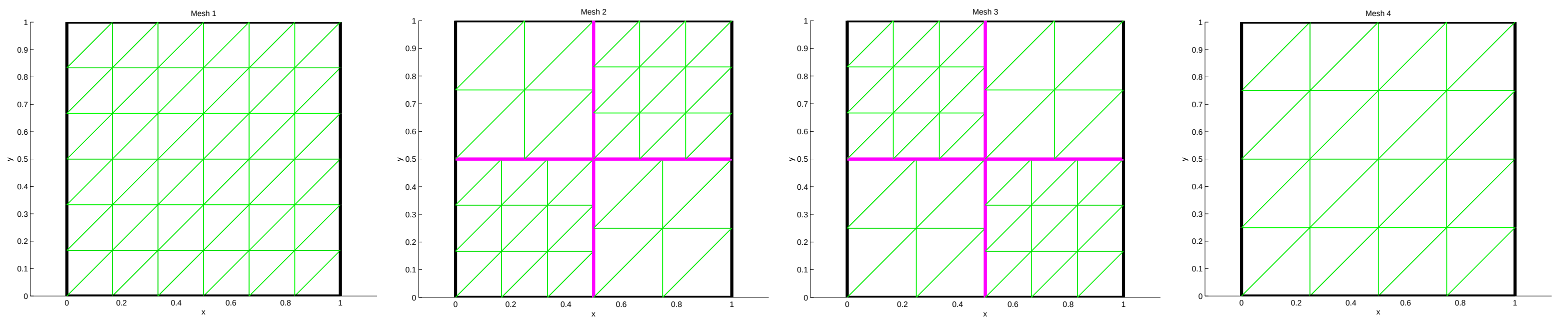
$\text{Error } |u - u_h|$ mesh size $h = 0.05$

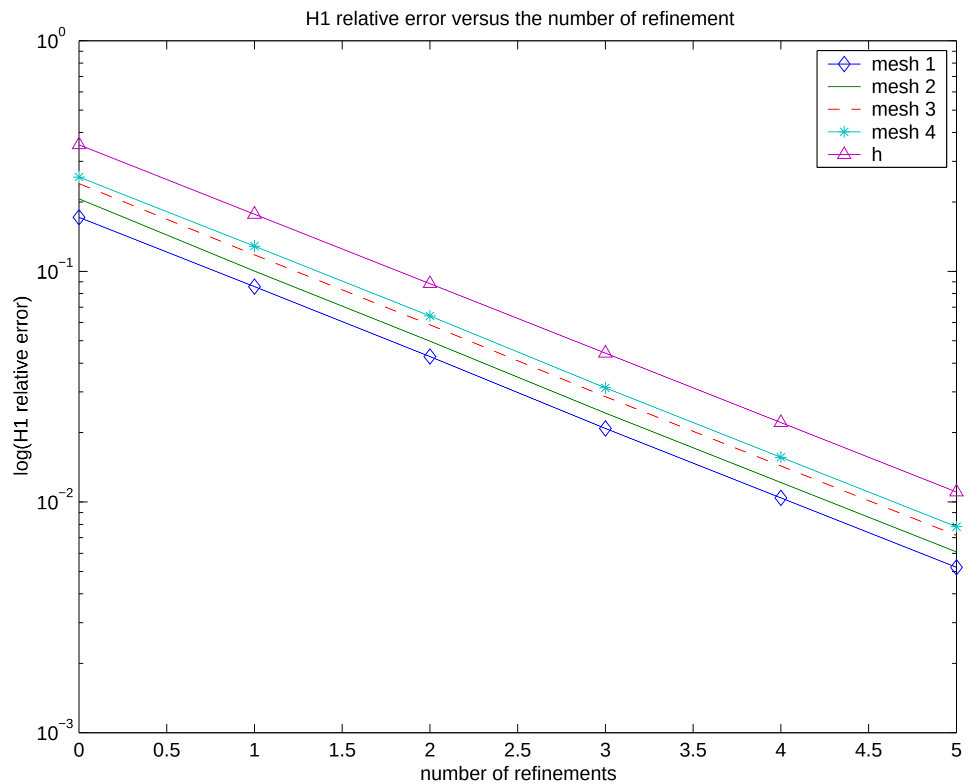


Error $|u - u_h|$ mesh size $h = 0.025$



H^1 relative error versus the number of refinements





Optimal Robin parameter

Conforming 2 subdomains case, with constant mesh size h :

$$\alpha_{opt} = [(\pi^2 + 1)((\frac{\pi}{h})^2 + 1)]^{\frac{1}{4}}.$$

Nonconforming case : 3 possibles choices for α :

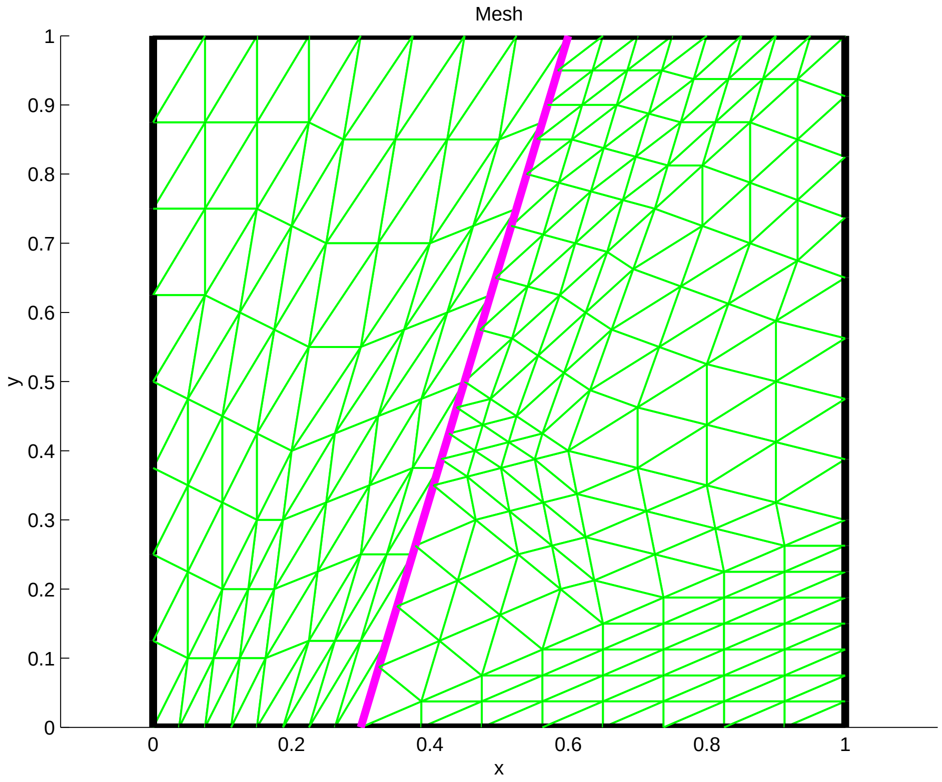
$$\alpha_{hmin} = [(\pi^2 + 1)((\frac{\pi}{h_{min}})^2 + 1)]^{\frac{1}{4}},$$

$$\alpha_{hmean} = [(\pi^2 + 1)((\frac{\pi}{h_{mean}})^2 + 1)]^{\frac{1}{4}},$$

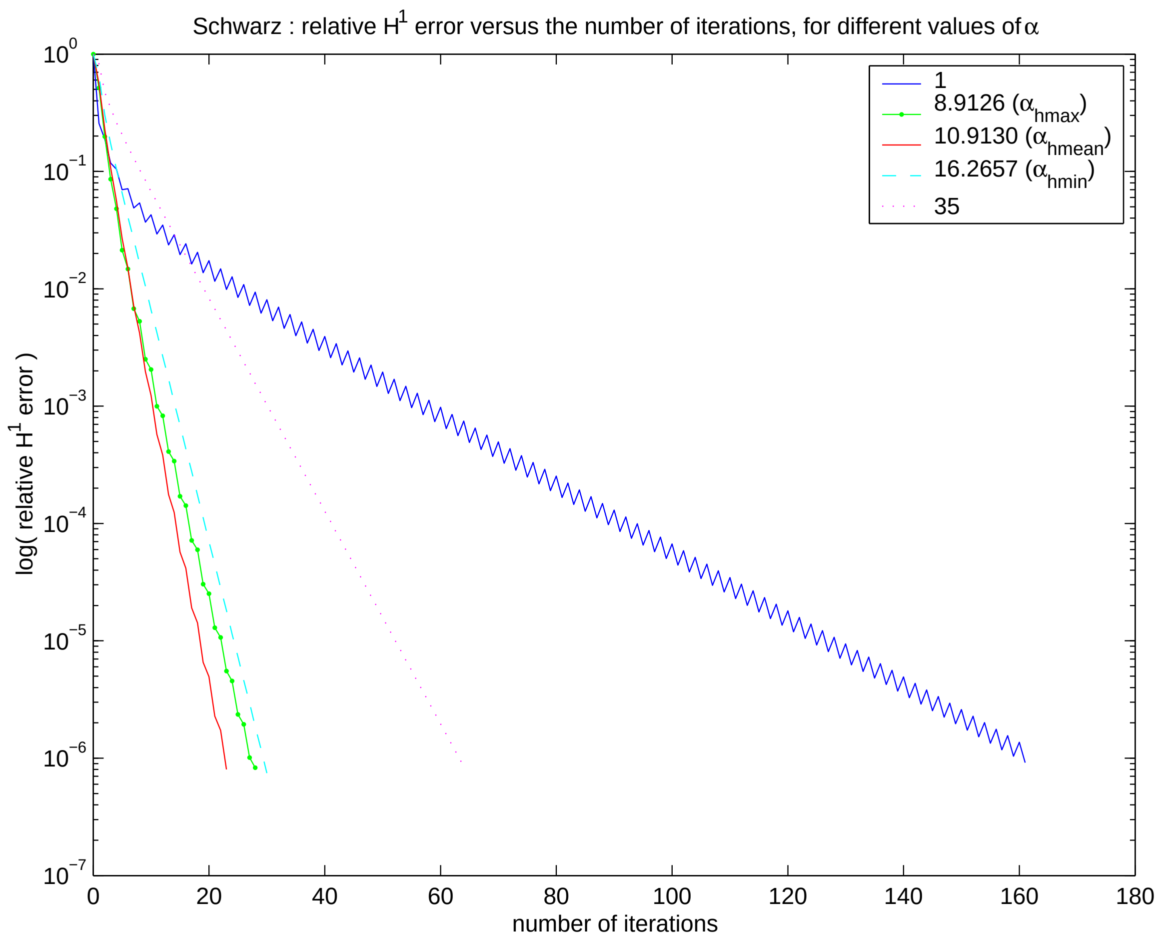
$$\alpha_{hmax} = [(\pi^2 + 1)((\frac{\pi}{h_{max}})^2 + 1)]^{\frac{1}{4}},$$

where h_{min}, h_{mean} and h_{max} are respectively the highest, mean, or lowest mesh size on the l'interface

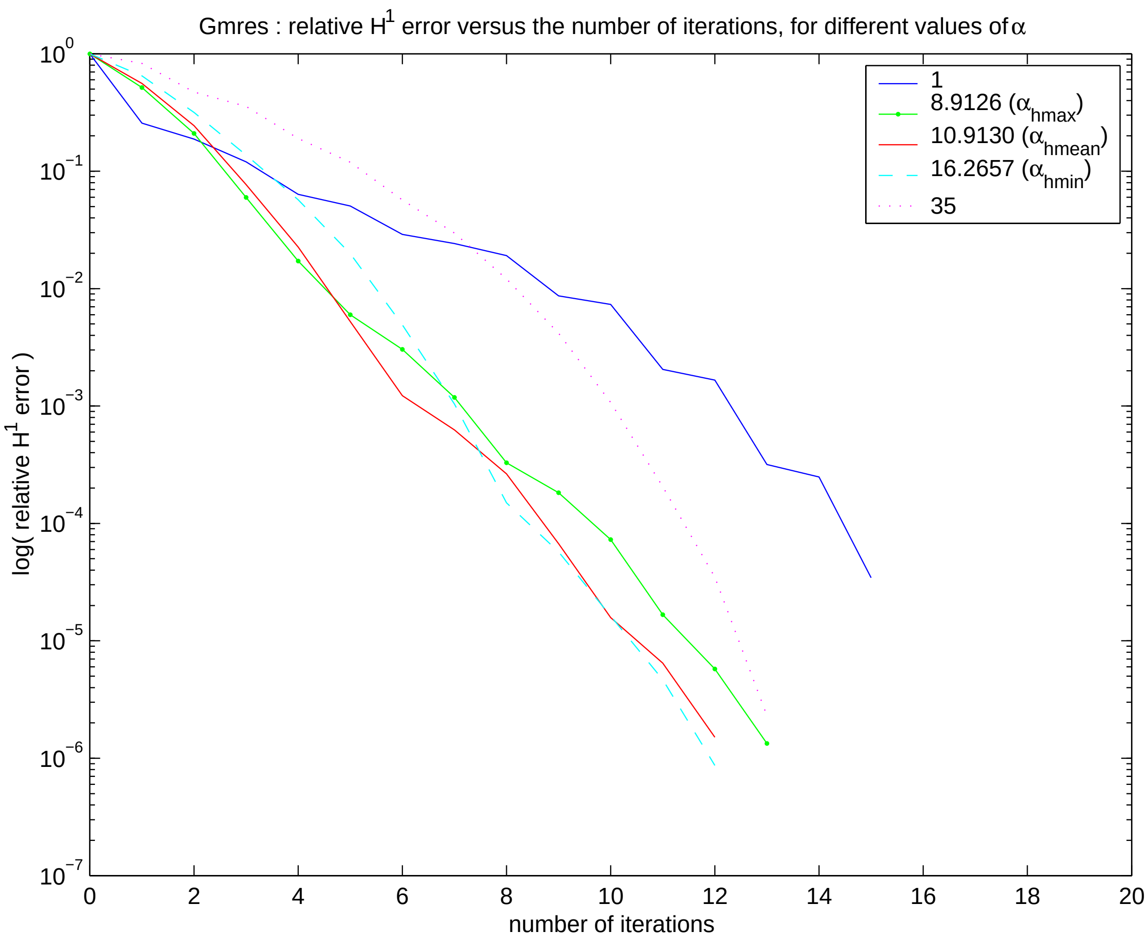
Nonconforming 2 subdomains decomposition



H^1 relative error versus the number of iterations for different values of α



H^1 relative error versus the number of iterations for different value of α (Gmres algorithm)



The algorithm is less sensitive to the choice of the Robin parameter.

Conclusions and Prospects

Schwarz type method with

- **Optimized Robin transmission conditions** $\rightarrow$ fast solver
- **Non-conforming grids** $\rightarrow$ independent meshes in the subdomains, optimal error estimates

$\Rightarrow$ Extension to Optimized Order 2 transmission conditions (with Y. Maday, F. Nataf)

$\Rightarrow$ Projection algorithm in 3D (with M.J. Gander)

$\Rightarrow$ Applications in 3D

$\Rightarrow$ Extension to ocean-atmospher coupling (with E. Blayo, L. Halpern)